Erratum - The analogue of Büchi's problem for rational functions

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The proof of Lemma 2.21 (ii) has a mistake (p. 561 §4) in the case where the x_n are polynomials in $F[t]$, with F of positive characteristic. It turns out that the statement of Theorem 1.5 is also incorrect in that situation. Theorem 1.8, its corollaries and their proofs are not affected by this mistake, but some changes in the proofs are required.

The enumeration of our statements corresponds to the enumeration of [1]. In this Erratum we provide the correct statement and the proof of Theorem 1.5. What changes is the case of characteristic $p \geq 0$. We want to thank Hector Pasten for pointing out the mistake to us, and for providing the following counter-example: if F is a field of characteristic $p \geq 3$ and $f \in F[t]$ then, for any $s \geq 1$, the sequence of squares of the sequence

$$x_n = (f+n)^{\frac{p^s+1}{2}}$$

has second difference equal to the constant sequence (2). We prove it below. What we prove here amounts to the fact that these are the only "non-trivial" sequences with this property.

We recall: we consider the following system of equations:

$$x_n^2 + x_{n-2}^2 = 2x_{n-1}^2 + 2 \qquad n = 2, \ldots, M-1 \tag{1}$$

over $F(t)$ where F is a field of characteristic either 0 or ≥ 3 and t is a variable.

We will call a sequence (x_n) a *Büchi sequence* if the second difference of the sequence (x_n^2) is constant and equal to 2, that is if the sequence (x_n) satisfies the System of Equations (1). We will call a Büchi sequence (x_n) *trivial* if for all n we have $x_n^2 = (x+n)^2$ for some $x \in F(t)$.

Theorem 1.5 should be:

Theorem 1.5 *Let F be a field and t a variable. Assume that $(x_n)_{n=0}^{M-1}$ is a Büchi sequence of rational functions $x_n \in F(t)$, not all constant.*

1. *Assume that one of the following holds:*
   (a) *The terms x_n of the sequence are polynomials (i.e. in $F[t]$), the characteristic of F is 0 and $M \geq 14$.*
   (b) *The characteristic of F is 0 and $M \geq 18$.*
   (c) *The characteristic of F is $p \geq 19$ and $M \geq 18$ and the following condition is not true:*
   Condition P: *there exists a positive integer s and an $f \in F[t]$ such that*
   $$x_n = (f+n)^{\frac{p^s+1}{2}}. \tag{P}$$
   (d) *The characteristic of F is $p \geq 17$ and $M \geq 14$ and all the x_n are polynomials in $F[t]$ and Condition P of 1(c) above is not true.*

   Then there are $\varepsilon_0, \ldots, \varepsilon_{M-1}$ with $\varepsilon_n \in \{-1, 1\}$ such that for each n, $\varepsilon_n x_n = \varepsilon_0 x_0 + n$.

2. *Assume that F has characteristic $p \geq 3$ and $f \in F[t]$. Then, for any $s \geq 1$, the sequence*
   $$x_n = (f+n)^{\frac{p^s+1}{2}}$$
   is a Büchi sequence.

Note that the new Theorem 1.5 provides a complete characterization of long enough Büchi sequences while the one in [1] did not. What is new is that the Büchi equation (1) in positive characteristic can have the solutions which satisfy Condition P of the Theorem; for long enough sequences there are no more solutions that those that satisfy Condition P and the tivial ones.

Lemma 2.21 from [1] has to be read as (we changed only the item (ii), removing the case where F has positive characteristic):

Lemma 2.21 *With assumptions and notation as in Lemma 2.16 we have:*
(i) If Equation (6) holds for one index then it holds for each index and there is a $\gamma \in F(t)$ such that for each index n we have $\gamma_n = \gamma + n$.
(ii) With the additional assumption that the characteristic of F is 0, the following holds: Equation (6) can not hold for any index.

The part of the proof of this Lemma that appears on p. 561 should be removed.

Proof of Item 1(c) and 1(d) of the new Theorem 1.5:

All Equations and Lemmas numbers given here refer to [1]. We will introduce symbols ($\star$), ($\dagger$) and ($\dagger\dagger$) for the Equations that did not appear in [1].

Let us recall that if $n \neq m$ then the quantity

$$\nu = \frac{x_n^2 - x_m^2}{n - m} - n - m$$

does not depend on n and m, and that ν_k denotes $\nu + 2k$ (see Definition 2.8).

The following lemma was implicitly proved in [1], but not stated in the way we need it for our current purposes, so we will supply a detailed proof of it.

Lemma A *Let F be an algebraically closed field of positive characterisitc p and let (x_n) be a non-trivial Büchi sequence of rational functions in $F(t)$ where not all x_n are p-th powers. If F has characteristic $p \geq 19$ and the sequence has 18 or more terms, or if F has characteristic $p \geq 17$ and the sequence (x_n) is a sequence of polynomials with 14 or more terms, respectively, then there exists $\gamma \in F(t)$ or $\gamma \in F[t]$, respectively, such that $\gamma' = 0$ and*

$$x_n^2 = (\nu - \gamma + n)(\gamma + n) = \left(\frac{\nu}{2} + n\right)^2 - \left(\frac{\nu}{2} - \gamma\right)^2$$

where $\nu = x_1^2 - x_0^2 - 1$ denotes the ν-invariant of the sequence (x_n).

Lemma A *Let F be an algebraically closed field of characteristic $p \geq 19$ (respectively, $p \geq 17$). Any non-trivial Büchi sequence (x_n) of rational functions in $F(t)$ with 18 or more terms (resp. of polynomials in $F[t]$ with 14 or more terms), where not all x_n are p-th powers, is such that*

$$x_n^2 = (\nu - \gamma + n)(\gamma + n) = \left(\frac{\nu}{2} + n\right)^2 - \left(\frac{\nu}{2} - \gamma\right)^2$$

for some $\gamma \in F(t)$ (resp. $\gamma \in F[t]$) such that $\gamma' = 0$, where $\nu = x_1^2 - x_0^2 - 1$ denotes the ν-invariant of the sequence (x_n).

Proof: From Lemma 2.18, we know that for each index n, one of the following two statements is true:

(a) There is a rational function $\gamma_n \in F(t) \setminus \{0\}$ such that $\gamma_n' = 0$ and

$$\nu_n = \frac{x_n^2}{\gamma_n} + \gamma_n \tag{6}$$

(b) There is a rational function δ_n such that $\delta_n' = 0$ and

$$\nu_n = 2\epsilon_n x_n + \delta_n \tag{7}$$

where $\epsilon_n \in \{1, -1\}$. From Corollary 2.20, if Equation (7) holds for some index n then it holds for all indices and the sequence (x_n) is the trivial sequence, contradicting our hypothesis. Hence, Equation (6) holds for each index n.

By Lemma 2.21 (i), there is a $\gamma \in F(t)$ such that for each index n we have $\gamma_n = \gamma + n$. The rest of the proof is contained in the correct part of the proof of Lemma 2.21 (ii). We reprove it here for the sake of completeness.

Observe that from Equation (6) it follows that

$$\nu - \gamma = \frac{x_m^2}{\gamma + m} - m$$

hence for all m we have

$$x_m^2 = (\nu - \gamma + m)(\gamma + m).$$

Note that if for some index k Equation (6) holds and $x_k \in F(t^p)$ then $\nu \in F(t^p)$ and then all x_n are in $F(t^p)$, which contradicts our hypothesis. Therefore we have $x_n \notin F(t^p)$ for each index n.

If all x_n are in $F[t]$ then, by definition of ν we have $\nu \in F[t]$ and since

$$x_m^2 = (\nu - \gamma + m)(\gamma + m) = (\nu + 2m - (\gamma + m))(\gamma + m)$$

we have

$$\left(\gamma + m - \frac{\nu + 2m}{2}\right)^2 = \frac{(\nu + 2m)^2}{4} - x_m^2$$

therefore $\gamma \in F[t^p]$. ◇

Let us now prove Theorem 1.5 under the hypothesis of clause 1(c) or 1(d):

Let (x_n) be a non-trivial Büchi sequence with 18 terms (14 if the x_n are polynomials), such that not all x_n are constant and at least one is not a p-th power. Applying Lemma A, we write $\gamma = f^{p^s}$, so that $f \in F[z] \setminus F[z^p]$ and for all n we have

$$\begin{aligned} x_n^2 &= (\nu - f^{p^s} + n)(f^{p^s} + n) \\ &= (\nu - f^{p^s} + n)(f + n)^{p^s} \end{aligned}$$

(note that if the x_n are polynomials then it follows that $(f+n)^{p^s-1}$ divides x_n^2). Considering the sequence of rational functions (or polynomials in the polynomial case) defined by

$$y_n = \frac{x_n}{(f+n)^{\frac{p^s-1}{2}}}$$

we have

$$\begin{aligned} y_n^2 &= (\nu - f^{p^s} + n)(f + n) \\ &= \left(\frac{\nu - f^{p^s} + f}{2} + n\right)^2 - \left(\frac{\nu - f^{p^s} + f}{2} - f\right)^2 \end{aligned}$$

hence

$$y_n^2 = \left(\frac{\bar{\nu}}{2} + n\right)^2 - \left(\frac{\bar{\nu}}{2} - f\right)^2 \qquad (\star)$$

where

$$\bar{\nu} = \nu - f^{p^s} + f. \qquad (\dagger)$$

We deduce that the sequence (y_n) is a Büchi sequence whose ν-invariant is $\bar{\nu}$ (the verification is easy and left to the reader).

It could be the case that y_n is a p-th power for each n. So we consider the sequence whose general term is z_n such that $y_n = z_n^{p^r}$ for some non-negative integer r such that not all z_n are p-th powers. It is easy to see that z_n is a Büchi sequence and that its ν-invariant is

$$\tilde{\nu} = \bar{\nu}^{\frac{1}{p^r}}.$$

We assume that (z_n) is a non-trivial Büchi sequence and will obtain a contradiction. Applying Lemma A to the sequence (z_n), there exists a p-th power $\tilde{\gamma}$ such that for all n

$$z_n^2 = \left(\frac{\tilde{\nu}}{2} + n\right)^2 - \left(\frac{\tilde{\nu}}{2} - \tilde{\gamma}\right)^2.$$

Hence we have

$$y_n^2 = (z_n^2)^{p^r} = \left(\frac{\bar{\nu}}{2} + n\right)^2 - \left(\frac{\bar{\nu}}{2} - \tilde{\gamma}^{p^r}\right)^2. \qquad (\dagger\dagger)$$

Comparing the two expressions of y_n^2 in Equations $\star$ and $\dagger\dagger$, we deduce

$$\left(\frac{\bar{\nu}}{2} - f\right)^2 = \left(\frac{\bar{\nu}}{2} - \tilde{\gamma}^{p^r}\right)^2.$$

Since f is not a p-th power, we have $f \neq \tilde{\gamma}^{p^r}$, hence

$$\frac{\bar{\nu}}{2} - f = -\frac{\bar{\nu}}{2} + \tilde{\gamma}^{p^r}$$

therefore,

$$\bar{\nu} = f + \tilde{\gamma}^{p^r}.$$

From Equation ($\dagger$) we deduce

$$\nu - f^{p^s} = \bar{\nu} - f = \tilde{\gamma}^{p^r}.$$

It follows that ν is a p-th power. Therefore,

$$\begin{aligned} x_n^2 &= (\nu - f^{p^s} + n)(f+n)^{p^s} \\ &= (\tilde{\gamma}^{p^r} + n)(f+n)^{p^s} \end{aligned}$$

is a p-th power, hence also each x_n is a p-th power. This gives a contradiction, hence proving that the sequence (z_n) is a trivial solution.

Since (z_n) is a trivial Büchi sequence, also $(y_n) = (z_n^{p^r})$ is a trivial Büchi sequence. By definition of the ν-invariant of the sequence (y_n), we have

$$\begin{aligned} \bar{\nu} &= \frac{y_n^2 - y_0^2}{n} - n \\ &= \frac{(y_0+n)^2 - y_0^2}{n} - n \end{aligned}$$

hence

$$\bar{\nu} = 2y_0.$$

Therefore, Equation ($\star$) becomes

$$(y_0 + n)^2 = (y_0 + n)^2 - (y_0 - f)^2$$

which implies $y_0 = f$, hence $\bar{\nu} = 2y_0 = 2f$. Equation (†) thus becomes

$$\nu = f^{p^s} + f$$

and

$$\begin{aligned} x_n^2 &= (\nu - f^{p^s} + n)(f + n)^{p^s} \\ &= (f^{p^s} + f - f^{p^s} + n)(f + n)^{p^s} \\ &= (f + n)^{p^s+1}. \end{aligned}$$

If the x_n are all p-th powers, we may consider the sequence (w_n) such that for each n we have $x_n = w_n^{p^r}$ and not all w_n are p-th powers. So we may apply the above argument to the sequence (w_n) and deduce that either (w_n) is such that $w_n^2 = (w + n)^2$ for some $w \in F(t)$ ($w \in F[t]$ if the x_n are polynomials), or there exists $f \in F(t)$ ($f \in F[t]$ if $x_n \in F[t]$) and a non-negative integer s such that $w_n^2 = (f + n)^{p^s+1}$. Therefore, either $x_n^2 = (w^{p^r} + n)^2$ is a trivial Büchi sequence, or

$$\begin{aligned} x_n &= \left[(f + n)^{\frac{p^s+1}{2}}\right]^{p^r} \\ &= (f^{p^r} + n)^{\frac{p^s+1}{2}}. \end{aligned}$$

We have proved 1(c) and (d) of the Theorem. It remains to prove (2), that is, to verify that if the sequence (x_n) satisfies Equations (P) then they are indeed Büchi sequences. Suppose that for each n we have

$$x_n = (f + n)^{\frac{p^s+1}{2}}$$

for some $f \in F[t]$ and s a non-negative integer. Then we obtain

$$\begin{aligned} x_n^2 &= (f + n)^{p^s+1} \\ &= (f + n)^{p^s}(f + n) \\ &= (f^{p^s} + n)(f + n) \\ &= \left(\frac{f^{p^s} + f}{2} + n\right)^2 - \left(\frac{f^{p^s} - f}{2}\right)^2 \end{aligned}$$

which has the form $(x + n)^2 + a$ for some polynomials x and a not depending on n. Therefore, the sequence (x_n) is a Büchi sequence: clearly, the second difference of a sequence of the form $((x+n)^2 - a)$ is the constant sequence (2).

Corrections in the Proof of Theorem 1.8 :

The conclusion of Theorem 1.8 remains correct, with minor changes in the proofs. Those are :

On page 562, case (A) of Proposition 3.1 should be:

(A) the hypothesis of case (i) of Theorem 1.8 holds, so either each x_i is in $F[t]$, and $z, w \in F$ or there is an $f \in F[t]$ and a positive integer s such that $w = f^{p^s+1}$ and $2z = f^{p^s} + f$.

We rewrite the proof of (i) of Theorem 1.8 (the last two paragraph of page 562). It should be :

(i) This is the case in which $R \subset F[t]$.
Consider the formula

$$\eta(z,w)\colon \phi(z,w) \wedge \phi(tz, t^2 w) \ .$$

We claim that it is equivalent to $w = z^2$. Clearly, it suffices to assume that not both z and w are equal to 0, and we adopt this as part of the assumption. It is obvious that any of the conditions $w = z^2$, and $t^2 w = (tz)^2$ implies the other. Hence, by Proposition 3.1 and Theorem 1.5, if $w \neq z^2$, we have that condition (A) holds for each of the pairs (z, w) and $(tz, t^2 w)$, hence one of the following holds :

(i1) z, $w \in F$ and tz, $t^2 w \in F$.

(i2) There are $f, g \in F[t]$ and positive integers s, r such that the following hold :

- $2z = f^{p^s} + f$,
- $w = f^{p^s+1}$,
- $2tz = g^{p^r} + g$,
- $t^2 w = g^{p^r+1}$,

It is obvious that (i1) is impossible. Hence case (i2) holds. We obtain that

$$t(f^{p^s} + f) = 2tz = g^{p^r} + g$$

which is impossible if $f^{p^s} + f$ is different from 0, since, otherwise, the degree of the left-hand side is 1 modulo p while the right-hand side either is 0 or has degree which is congruent to 0 modulo p. We conclude that $f^{p^s} + f = 0$ and $g^{p^r} + g = 0$. Hence $z = 0$ and both w and $t^2 w$ are elements of F; it follows that $w = 0$, which contradicts our hypothesis. This concludes the proof.

References

[1] T. Pheidas and X. Vidaux, *The analogue of Büchi's problem for rational functions*, Journal of The London Mathematical Society **74-3**, 545-565 (2006).