Electromagnetism: Second homework

Oscar Castillo-Felisola
Prof.: Crisóstomo

May 14, 2005

First Exercise

Let us consider the generators of the Lorentz boost and rotations in a 4-dimensional space-time, and call them $\vec{K}$ and $\vec{S}$. They are represented by:

$$K^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$S^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad S^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad S^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using the above representation of the generators of the Lorentz group, we can write

$$\vec{\epsilon} \cdot \vec{S} = \begin{pmatrix} 0 & 0 & 0 & \\ 0 & 0 & \epsilon_3 & \epsilon_2 \\ 0 & -\epsilon_3 & 0 & \epsilon_1 \\ 0 & -\epsilon_2 & -\epsilon_1 & 0 \end{pmatrix}, \tag{1}$$

and the standard matrix multiplication yields,

$$(\vec{\epsilon} \cdot \vec{S})^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\epsilon_3^2 - \epsilon_2^2 & -\epsilon_1 \epsilon_2 & \epsilon_3 \epsilon_1 \\ 0 & -\epsilon_2 \epsilon_1 & -\epsilon_3^2 \epsilon_1^2 & -\epsilon_3 \epsilon_2 \\ 0 & \epsilon_3 \epsilon_1 & -\epsilon_3 \epsilon_2 & -\epsilon_2^2 - \epsilon_1^2 \end{pmatrix}, \tag{2}$$

and finally,

$$(\vec{\epsilon} \cdot \vec{S})^3 = |\epsilon|^2 \begin{pmatrix} 0 & 0 & 0 & \\ 0 & 0 & \epsilon_3 & \epsilon_2 \\ 0 & -\epsilon_3 & 0 & \epsilon_1 \\ 0 & -\epsilon_2 & -\epsilon_1 & 0 \end{pmatrix}. \tag{3}$$

Similarly, for the $\vec{K}$ generators, we get the following

$$\vec{\epsilon}\cdot\vec{K} = \begin{pmatrix} 0 & \epsilon_1 & \epsilon_2 & \epsilon_3 \\ \epsilon_1 & 0 & 0 & 0 \\ \epsilon_2 & 0 & 0 & 0 \\ \epsilon_3 & 0 & 0 & 0 \end{pmatrix}, \tag{4}$$

$$(\vec{\epsilon}\cdot\vec{K})^2 = \begin{pmatrix} |\epsilon|^2 & 0 & 0 & 0 \\ 0 & \epsilon_1^2 & \epsilon_1\epsilon_2 & \epsilon_1\epsilon_3 \\ 0 & \epsilon_1\epsilon_2 & \epsilon_2^2 & \epsilon_2\epsilon_3 \\ 0 & \epsilon_3\epsilon_1 & \epsilon_2\epsilon_3 & \epsilon_3^2 \end{pmatrix},$$

and

$$(\vec{\epsilon}\cdot\vec{K})^3 = |\epsilon|^2 \begin{pmatrix} 0 & \epsilon_1 & \epsilon_2 & \epsilon_3 \\ \epsilon_1 & 0 & 0 & 0 \\ \epsilon_2 & 0 & 0 & 0 \\ \epsilon_3 & 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

For the following part, let's assume that $|\epsilon|^2 = 1$, so $(\vec{\epsilon}\cdot\vec{K})^{2n+1} = \vec{\epsilon}\cdot\vec{K}\ \forall\ n \in \mathbb{N}$ and $(\vec{\epsilon}\cdot\vec{K})^{2n} = (\vec{\epsilon}\cdot\vec{K})^2\ \forall\ n \in \mathbb{N}$. Thus,

$$\begin{aligned}
e^{-\zeta\vec{\beta}\cdot\vec{K}} &= \sum_{n=0}^{\infty} \frac{(-1)^n\zeta^n}{n!}(\vec{\beta}\cdot\vec{K})^n \\
&= \sum_{n=even} \frac{\zeta^n}{n!}(\vec{\beta}\cdot\vec{K})^n - \sum_{n=odd} \frac{\zeta^n}{n!}(\vec{\beta}\cdot\vec{K})^n \\
&= 1 + \sum_{n=1}^{\infty} \frac{\zeta^{2n}}{2n!}(\vec{\beta}\cdot\vec{K})^{2n} - \sum_{n=0}^{\infty} \frac{\zeta^{2n+1}}{(2n+1)!}(\vec{\beta}\cdot\vec{K})^{2n+1} \\
&= 1 + (\vec{\beta}\cdot\vec{K})^2 \sum_{n=1}^{\infty} \frac{\zeta^{2n}}{2n!} - (\vec{\beta}\cdot\vec{K}) \sum_{n=0}^{\infty} \frac{\zeta^{2n+1}}{(2n+1)!} \\
&= 1 - (\vec{\beta}\cdot\vec{K})\sinh(\zeta) + (\vec{\beta}\cdot\vec{K})^2(\cosh(\zeta) - 1). \qquad (6)
\end{aligned}$$

Second Exercise

Let $A_1 = e^L$ and $A_2 = e^{-L+\delta}$ be operators[1], we want to compute $A = A_1A_2$. Since A's are operators, we should understand them as the series expansion

---

[1] If the second operator has not the minus sign in the L term, the series contains other non-vanishing terms linear on δ

of the operator. So, we have (by taking off all term of order grander than δ^2)

$$\begin{aligned}
e^L e^{-L+\delta} &= \sum_{n=0}^{\infty} \frac{L^n}{n!} \sum_{m=0}^{\infty} \frac{(-L+\delta)^m}{m!} \\
&= \left(1 + L + \frac{L^2}{2!} + \frac{L^3}{3!} + \frac{L^4}{4!} + \cdots\right)\left(1 - L + \delta + \frac{(-L+\delta)^2}{2!} + \frac{(-L+\delta)^3}{3!} + \frac{(-L+\delta)^4}{4!} + \cdots\right) \\
&= 1 + \delta - \frac{L\delta + \delta L}{2!} + L\delta + \frac{L^2\delta + L\delta L + \delta L^2}{3!} - \frac{L^2\delta + L\delta L}{2!} + \frac{L^2\delta}{2!} \\
&\quad - \frac{L^3\delta + L^2\delta L + L\delta L^2 + \delta L^3}{4!} + \frac{L^3\delta + L^2\delta L + L\delta L^2}{3!} - \frac{L^3\delta + L^2\delta L}{2!2!} + \frac{L^3\delta}{3!} + \cdots \\
&= 1 + \delta + \frac{L\delta - \delta L}{2!} + \frac{L^2\delta - 2L\delta L + \delta L^2}{3!} + \frac{L^3\delta - 3L^2\delta L - 3L\delta L^2 + \delta L^3}{4!} + \cdots \\
&= 1 + \delta + \frac{1}{2!}[L,\delta] + \frac{1}{3!}[L,[L,\delta]] + \frac{1}{4!}[L,[L,[L,\delta]]] + \cdots \qquad (7)
\end{aligned}$$

Third Exercise

In a 2-dimensional inelastic collision between two particles, the "4"-momentum is still conserved, so

$$p_\mu^{(1)} + p_\mu^{(2)} = p_\mu^{(f)}, \qquad (8)$$

where

$$p_\mu^{(1)} = m_1(\gamma_1, u_1\gamma_1) \quad \text{and} \quad p_\mu^{(2)} = m_2(\gamma_2, u_2\gamma_2), \qquad (9)$$

then

$$p_\mu^{(f)} = (m_1\gamma_1 + m_2\gamma_2, m_1u_1\gamma_1 + m_2u_2\gamma_2). \qquad (10)$$

We can finding the mass of the final particle by compute the invariant quantity $p_\mu^{(f)}p^{\mu(f)}$, so

$$\begin{aligned}
-M^2 &= p_\mu^{(f)}p^{\mu(f)} \\
&= -(m_1\gamma_1 + m_2\gamma_2)^2 + (m_1u_1\gamma_1 + m_2u_2\gamma_2)^2 \\
&= -m_1^2 - m_2^2 - 2m_1m_2\gamma_1\gamma_2(1 - u_1u_2). \\
\Longrightarrow \quad M^2 &= m_1^2 + m_2^2 + 2m_1m_2\gamma_1\gamma_2(1 - u_1u_2). \qquad (11)
\end{aligned}$$

Even, the velocity is

$$u = \frac{p_1^{(f)}}{p_0^{(f)}} = \frac{m_1u_1\gamma_1 + m_2u_2\gamma_2}{m_1\gamma_1 + m_2\gamma_2}. \qquad (12)$$

Fourth Exercise

Now, we shall consider a 2-dimensional elastic collision between two particles, one of them with mass m_0 and the other with mass M. Given the initial condition of our system, we can write down the "4"-momentum of the particles, as follows

$$p_\mu^{(1)} = (e_0, p_0), \qquad p_\mu^{(2)} = (M, 0), \qquad (13)$$

$$p_\mu^{(3)} = (e, p), \qquad p_\mu^{(4)} = (E, P), \qquad (14)$$

From the conservation of the 4-momentum, we get

$$\begin{aligned} p_\mu^{(1)} + p_\mu^{(2)} &= p_\mu^{(3)} + p_\mu^{(4)} \\ p_\mu^{(1)} + p_\mu^{(2)} - p_\mu^{(4)} &= p_\mu^{(3)}. \end{aligned} \tag{15}$$

Then,

$$p_\mu^{(1)} p^{\mu(1)} + p_\mu^{(2)} p^{\mu(2)} + p_\mu^{(4)} p^{\mu(4)} + 2p_\mu^{(1)} p^{\mu(2)} - 2p_\mu^{(1)} p^{\mu(4)} - 2p_\mu^{(2)} p^{\mu(4)} = p_\mu^{(3)} p^{\mu(3)}. \tag{16}$$

Since

$$p_\mu^{(1)} p^{\mu(1)} = p_\mu^{(3)} p^{\mu(3)} = -m_0^2 \quad \text{and} \quad p_\mu^{(1)} p^{\mu(1)} = p_\mu^{(1)} p^{\mu(1)} = -M^2, \tag{17}$$

we have

$$-M^2 - e_0 M + e_0 E - p_0 P + EM = 0, \tag{18}$$

$$E(M + e_0) = M^2 + e_0 M + p_0 P. \tag{19}$$

As we know that $E = \sqrt{M^2 + P^2}$

$$\begin{aligned} (M^2 + P^2)(M + e_0)^2 &= M^2 (M + e_0)^2 + p_0^2 P^2 + 2p_0 PM(M + e_0) \\ P\left((M + e_0)^2 - p_0^2\right) &= 2p_0 M(M + e_0) \\ P &= \frac{2p_0 M(M + e_0)}{M^2 + 2Me_0 + m_0^2}. \end{aligned} \tag{20}$$

Finally, by using the conservation of the 4-momentum, $p_0 = p + P$, we obtain

$$p = \frac{p_0(m_0^2 - M^2)}{M^2 + 2Me_0 + m_0^2}. \tag{21}$$