Exam of Electrodynamics

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Contents

1 Problem 11.6

In this problem, we shall use the relativistic addition of velocities. Let's consider the 2 velocities, u and v, where u is the velocity of the fluid respects to the laboratory and v is the velocity of light in the medium, i. e., $v = \frac{c}{n}$, with c the speed of light in vacuum and n the refraction index.

By adding the velocities, the velocity of light respect to the lab frame is

$$u' = \frac{u+v}{1+\frac{uv}{c^2}}. \tag{1}$$

Now, as $u \ll 1$, we can use the approximation

$$\frac{1}{1-x} = 1 + x,$$

and getting

$$\begin{aligned} u' &= \frac{u+v}{1+\frac{uv}{c^2}} \\ &\sim (u+v)\left(1-\frac{uv}{c^2}\right) \\ &= v+u\left(1-\frac{v^2}{c^2}\right) \\ &= \frac{c}{n}+u\left(1-\frac{1}{n^2}\right). \end{aligned} \tag{2}$$

In the last expression, we got the velocity of light on the lab frame but using the refraction index in the fluid frame. Our next step should be write $n = n(n')$ and substitute that in (2).

As we now, $n = \frac{ck}{\omega}$, and by the Lorentz transformation of the phase, we have

$$\frac{\omega}{c} = \gamma\left(\frac{\omega'}{c}+\beta k'\right) \tag{3}$$

and

$$k = \gamma\left(\frac{\beta\omega'}{c}+k\right), \tag{4}$$

so,

$$\begin{aligned} n &= \frac{ck}{\omega} \\ &= \frac{k'+\frac{\beta\omega'}{c}}{\frac{\omega'}{c}+\beta k'} \\ &= n'\left(\frac{1+\frac{\beta}{n'}}{1+\beta n'}\right), \end{aligned} \tag{5}$$

but also from the Lorentz transformation its follows that

$$\omega = \gamma\left(\omega'+c\beta k'\right) = \gamma\omega'\left(1+\beta n'\right) \sim \omega'\left(1+\beta n'\right), \tag{6}$$

where we have used the fact that for $u \ll 1$, $\gamma \sim 1$. Thus,

$$\omega'-\omega \cong \beta\omega' n', \tag{7}$$

at least at first order.

Expanding n in its Taylor series (up to first order), we obtain

$$n = n' + \frac{dn'}{d\omega'}(\omega'-\omega) + \mathscr{O}(\beta^2) \tag{8}$$

or

$$n = n' - \beta\omega' n' \frac{dn'}{d\omega'}. \tag{9}$$

Finally, (2) becomes

$$\begin{aligned} u' &= \frac{c}{n} + u\left(1 - \frac{1}{n^2}\right) \\ &= \frac{c}{n' - \beta\omega' n' \frac{dn'}{d\omega'}} + u\left(1 - \frac{1}{\left(n' - \beta\omega' n' \frac{dn'}{d\omega'}\right)^2}\right) \\ &= \frac{c}{n'}\left(1 + \beta\omega' \frac{dn'}{d\omega'}\right) + u\left(1 - \frac{1}{n'^2}\left(1 + \beta\omega' \frac{dn'}{d\omega'}\right)\right), \end{aligned} \tag{10}$$

since $\beta = \frac{u}{c}$ is of order u, the term which contains $u\beta$ is dropped because $u \ll 1$, and we left just with

$$u' = \frac{c}{n'} + u\left(1 - \frac{1}{n'^2} + \frac{\omega'}{n'} \frac{dn'}{d\omega'}\right). \tag{11}$$

2 Problem 11.20

2.1 Part a

We know that at each reference frame, the 4-momentum (p_μ) is conserved, but also that in all the frames, the quantity $p^\mu p_\mu$ is the same.

Let us write the 4-momentum for the initial states in both cases. In the lab frame, we have

$$p_\mu^{(1)} = (E_{lab}, p_{lab}) \tag{12}$$
$$p_\mu^{(2)} = (m_2, 0). \tag{13}$$

And for the centre of mass frame, we have

$$p'^{(1)}_\mu = (E'_1, p') \tag{14}$$
$$p'^{(2)}_\mu = (E'_2, -p'). \tag{15}$$

So, the invariance of the momentum squares yields,

$$(E_{lab} + m_2, p_{lab})^2 = (E'_1 + E'_2, 0)^2 \tag{16}$$
$$-m_1^2 - m_2^2 - 2m_2 E_{lab} = -E'^2, \tag{17}$$

or

$$E'^2 = m_1^2 + m_2^2 + 2m_2 E_{lab}, \tag{18}$$

where $E' = E_1' + E_2'$ is the total energy in the centre of mass frame.

In the same way, from (16) we obtain

$$2m_2 E_{lab} = 2p'^2 + 2E_1'E_2', \tag{19}$$

by using the fact that $E^2 = m^2 + p^2$. Next, let's take the square of the above equation

$$\begin{aligned}
m_2^2 E_{lab}^2 + p'^4 - 2m_2 E_{lab} p'^2 &= m_1^2 m_2^2 + (m_1^2 + m_2^2)p'^2 + p'^4 \\
m_2^2(m_1^2 + p_{lab}^2) - 2m_2 E_{lab} p'^2 &= m_1^2 m_2^2 + (m_1^2 + m_2^2)p'^2 \\
\Rightarrow \quad p'^2 &= \frac{m_2^2 p_{lab}^2}{m_1^2 + m_2^2 + 2m_2 E_{lab}} \qquad (20)
\end{aligned}$$

or

$$p' = \frac{m_2 p_{lab}}{E'}. \tag{21}$$

2.2 Part b

From the definition of centre of mass,

$$\beta_{CM} = \frac{\sum p_{lab}}{\sum E_{lab}}.$$

It follows that

$$\beta_{CM} = \frac{p_{lab}}{E_{lab} + m_2}. \tag{22}$$

An also,

$$\begin{aligned}
\gamma &= \frac{1}{\sqrt{1-\beta^2}} \\
&= \frac{1}{\left(1 - \left(\frac{p_{lab}}{E_{lab}+m_2}\right)^2\right)^{\frac{1}{2}}} \\
&= \left[\frac{(E_{lab}+m_2)^2 - p_{lab}^2}{(E_{lab}+m_2)^2}\right]^{-\frac{1}{2}} \\
&= \frac{E_{lab}+m_2}{\sqrt{(E_{lab}+m_2)^2 - p_{lab}^2}}. \qquad (23)
\end{aligned}$$

By using (16) we get

$$\gamma = \frac{E_{lab} + m_2}{E'}. \tag{24}$$

2.3 Part c

Since

$$E'^2 = m_1^2 + m_2^2 + 2m_2 E_{lab}, \tag{25}$$

then

$$\begin{aligned}
E' &= \sqrt{m_1^2 + m_2^2 + 2m_2 E_{lab}} \\
&= (m_1 + m_2)\sqrt{1 + \frac{2m_2(E_{lab} - m_1)}{(m_1 + m_2)^2}} \\
&= (m_1 + m_2)\left(1 + \frac{m_2(E_{lab} - m_1)}{(m_1 + m_2)^2}\right) \\
&= m_1 + m_2 + \frac{m_2(E_{lab} - m_1)}{(m_1 + m_2)}. \qquad (26)
\end{aligned}$$

Expanding $E_{lab} = m_1 + \frac{p_{lab}^2}{2m_1} + \mathscr{O}(p^4)$, the above expression becomes,

$$E' = m_1 + m_2 + \frac{m_2}{(m_1 + m_2)} \frac{p_{lab}^2}{2m_1}. \tag{27}$$

Consequently,

$$\begin{aligned}
p' &= \frac{m_2 p_{lab}}{E'} \\
&= \frac{m_2 p_{lab}}{(m_1 + m_2)\left(1 + \frac{m_2}{(m_1+m_2)^2} \frac{p_{lab}^2}{2m_1}\right)} \\
&= \frac{m_2 p_{lab}}{(m_1 + m_2)} \left(1 - \frac{m_2}{(m_1 + m_2)^2} \frac{p_{lab}^2}{2m_1}\right) \\
&= \frac{m_2 p_{lab}}{(m_1 + m_2)} + \mathscr{O}(p^3), \qquad (28)
\end{aligned}$$

and

$$\begin{aligned}
\beta_{CM} &= \frac{p_{lab}}{m_2 + E_{lab}} \\
&= \frac{p_{lab}}{m_2 + \sqrt{m_1^2 + p_{lab}^2}} \\
&= \frac{p_{lab}}{m_2 + m_1}, \qquad (29)
\end{aligned}$$

3 Problem 12.2

3.1 Part a

The action is defined as

$$S = \int \mathscr{L} dt, \tag{30}$$

consider now that $\mathscr{L} \to \mathscr{L} + d_t\chi$, so

$$\begin{aligned} S &= \int \mathscr{L} dt + \int \frac{d}{dt}\chi dt \\ &= \int \mathscr{L} dt + \int d\chi, \end{aligned} \tag{31}$$

where we have used the chain rule in the last step.

Under variations,

$$\delta S = \int \left(\frac{\delta \mathscr{L}}{\delta q} - \partial_t \frac{\delta \mathscr{L}}{\delta \dot{q}} \right) + \delta(\chi(t_2) - \chi(t_1)), \tag{32}$$

but by the conditions of the variational problem, i. e., keeping fixed the end points (boundaries), it follows that both lagrangian yield the same Euler-Lagrange equations.

3.2 Part b

Let's consider the Lagrangian

$$\mathscr{L} = -m\sqrt{1 - v^2} + e\vec{v} \cdot \vec{A} - e\phi, \tag{33}$$

under the transformations $A_\mu \to A'_\mu = A_\mu + \partial_\mu \chi$. Thus,

$$\begin{aligned} \mathscr{L}' &= -m\sqrt{1 - v^2} + e\vec{v} \cdot \vec{A}' - e\phi' \qquad (34) \\ &= -m\sqrt{1 - v^2} + e\vec{v} \cdot (\vec{A} + \vec{\nabla}\chi) - e(\phi - \partial_t \chi), \qquad (35) \end{aligned}$$

by remember that $\partial^t = -\partial_t$. Next,

$$\mathscr{L}' = \mathscr{L} + e\vec{v} \cdot \vec{\nabla}\chi + e\partial_t \chi, \tag{36}$$

or, by using the chain rule, $\vec{v} \cdot \vec{\nabla} + \partial_t = d_t$, we can write the above equation as

$$\mathscr{L}' = \mathscr{L} + e\frac{d}{dt}\chi, \tag{37}$$

that by the argument of the part 3.1 both Lagrangian are equivalents.

4 Problem 12.7

4.1 Part b

Let's consider the Lorentz force for a particle of mass m and charge q, where the electric and magnetic fields are parellels, statics and uniforms.

The field strength for the system is given by

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & -B \\ 0 & 0 & B & 0 \end{pmatrix}, \tag{38}$$

so, the Lorentz force is given by the following system of equations

$$\frac{du^0}{d\tau} = -\frac{q}{m}Eu^1 \tag{39}$$

$$\frac{du^1}{d\tau} = -\frac{q}{m}Eu^0 \tag{40}$$

$$\frac{du^2}{d\tau} = -\frac{q}{m}Bu^3 \tag{41}$$

$$\frac{du^3}{d\tau} = \frac{q}{m}Bu^2. \tag{42}$$

Deriving (41) and substituting on (42) and conversely, we can decouple the equations for u^2 and u^3, getting

$$\frac{d^2u^2}{d\tau^2} + \omega^2 u^2 = 0, \tag{43}$$

and

$$\frac{d^2u^3}{d\tau^2} + \omega^2 u^3 = 0, \tag{44}$$

where $\omega = \frac{qB}{m}$, whose solutions are under a well choice of the coordinate are just

$$y \sim \sin(\omega\tau) \quad \text{and} \quad z \sim \cos(\omega\tau), \tag{45}$$

with the last choice of axes, one note that

$$y\frac{dy}{d\tau} + z\frac{dz}{d\tau} = 0, \tag{46}$$

thus,

$$y^2 + z^2 = R^2 = \text{constant}. \tag{47}$$

Other way of thought the above result is by parametrise the coordinates as follows,

$$y = R\sin(\omega\tau) \quad \text{and} \quad z = R\cos(\omega\tau). \tag{48}$$

In the same way, we can rearrange the equations (39) and (40), whose solutions are not other than hyperbolic sine and cosine,

$$t \sim \sinh(\lambda\tau) \quad \text{and} \quad x \sim \cosh(\lambda\tau), \tag{49}$$

with $\lambda = \frac{qE}{m}$. The eq. (49) can be written as,

$$x\frac{dx}{d\tau} - t\frac{dt}{d\tau} = 0, \tag{50}$$

or

$$x^2 - t^2 = P^2 = \text{constant}, \tag{51}$$

or similarly,

$$t = P\sinh(\lambda\tau) \quad \text{and} \quad x = P\cosh(\lambda\tau). \tag{52}$$

From now on, let's call $\phi = \omega\tau$, and from this, ot follows that $\lambda\tau = \frac{\lambda}{\omega}\phi = \rho\phi = \frac{E}{B}\phi$.

Now, we write down the 4-position and the 4-velocity

$$x^\mu = (t, x, y, z), \tag{53}$$

and

$$\begin{aligned} u^\mu &= (\lambda x, \lambda t, \omega x, -\omega y) \\ &= (P\lambda\cosh(\rho\phi), P\lambda\sinh(\rho\phi), \omega R\cos(\phi), -\omega R\sin(\phi)). \end{aligned} \tag{54}$$

From the condition $u^\mu u_\mu = -1$, we have that

$$P^2\lambda^2\cosh^2(\rho\phi) - P^2\lambda^2\sinh^2(\rho\phi) - \omega^2R^2\cos^2(\phi) - \omega^2R^2\sin^2(\phi) = 1, \tag{55}$$

that is nothing but

$$P^2\lambda^2 - \omega^2R^2 = 1, \tag{56}$$

which yields to

$$P = \frac{1}{\lambda}\sqrt{1 + \omega^2R^2}. \tag{57}$$

Finally,

$$y = R\sin(\phi) \tag{58}$$

$$z = R\cos(\phi) \tag{59}$$

$$x = \frac{1}{\lambda}\sqrt{1 + \omega^2R^2}\cosh(\rho\phi) \tag{60}$$

$$t = \frac{1}{\lambda}\sqrt{1 + \omega^2R^2}\sinh(\rho\phi) \tag{61}$$