Classical mechanics:
A first course

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Contents

Introduction

The aim of this notes is to give a self-contained introduction to classical mechanics, particularly lagrangian and hamiltonian mechanics, as well as conserved quantities and integration of the more elemental cases.

Clearly, the whole physical part of the notes are explained at the usual books on the subject, let's say Landau and Lifshitz or Goldstein.

In the first chapter we introduce the Lagrangian mechanics and we show how it generalizes the Newtonian way of studying the problem. Second chapter introduce the notion of conserved quantities and how they are obtained. Third chapter is rather important and most difficult (by the way), we learn ways to solve the equations of motion. Fourth chapter introduce the Hamiltonian mechanics.

Chapter 1

Lagrangian Mechanics

All of us are familiar with Newtonian way of doing mechanics, that is making use of the three Newton's laws:

- Inertia.
- $\vec{F} = m\vec{a}$.
- Action and reaction.

Indeed, this is correct, but just for certain systems. Now, we can ask ourselves, if it isn't the "correct" way of doing mechanics, What is the most general?

1.1 Action and Lagrangian

In order to avoid theological troubles, let's say "**Nature is lazy**", for this reason it minimizes its **action**.

By definition, action (S) is

$$S = \int_{t_1}^{t_2} \mathcal{L}(t, \vec{x}, \vec{v}), \tag{1.1}$$

where $\mathcal{L}$ is the Lagrangian, which is the kinetic energy minus the potential,

$$\mathcal{L} = \frac{1}{2} m\vec{v} \cdot \vec{v} - U(\vec{x}). \tag{1.2}$$

More generally, we'll use the called generalized coordinates and velocities, denoted by $\vec{q}(t)$ and $\dot{\vec{q}}(t)$ respectively. In (1.2) U just depends on $\vec{x}$ in order to rise instantaneously propagating interaction.

Lagrangians and actions are "functions" of function, these are called functionals. If functions and functionals are not the same, mostly we can forget this and try'em as functions.

Note that the Lagrangian is a scalar quantity, even if it's constructed from other geometrical objects.

1.2 Minimum Action

In order to get the equations of motion, we must minimize the action. This procedure is quit similar to the one of functions, except that we need a couple of boundary conditions.

We want to minimize the action between t_1 and t_2, and of course $\delta\vec{x}_i = 0$, so

$$\begin{aligned}
\delta S &= S(\vec{x}+\delta\vec{x}, \vec{v}+\delta\vec{v}) - S(\vec{x},\vec{v}) \\
&= \int_{t_1}^{t_2} dt \left(\frac{\partial \mathcal{L}}{\partial \vec{x}} \cdot \delta\vec{x} + \frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}} \cdot \delta\dot{\vec{x}} \right) \\
&= \int_{t_1}^{t_2} dt \left(\frac{\partial \mathcal{L}}{\partial \vec{x}} \cdot \delta\vec{x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}} \cdot \delta\vec{x} \right) + \left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}} \cdot \delta\vec{x} \right)\Bigg|_{t_1}^{t_2} \\
&= \int_{t_1}^{t_2} dt \left(\frac{\partial \mathcal{L}}{\partial \delta\vec{x}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}} \right) \cdot \delta\vec{x} \\
&= 0,
\end{aligned} \tag{1.3}$$

Then,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}} - \frac{\partial \mathcal{L}}{\partial \vec{x}} = 0, \tag{1.4}$$

gives us the equation of motion of the system.

Note that in a n-dimensional space, we have n independent equations of motion, i.e., one for each coordinate.

Equations (1.4) are called Euler-Lagrange equations, and each times we want to obtain the equations of a system we begin writing the lagrangian of the system and after apply the Euler-Lagrange equations.

The $\frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}} = \vec{p}$ are said to be the canonical momenta. If we are working in generalized coordinates, they are called the generalized canonical momenta.

Example 1 (Particle in a potential). *Let's consider a particle of mass m in a potential $U(\vec{x})$, then its lagrangian is*

$$\mathcal{L} = \frac{1}{2} m\vec{v}\cdot\vec{v} - U(\vec{x}), \tag{1.5}$$

therefore, from (1.4) we have,

$$\frac{d}{dt}(m\vec{v}) = -\vec{\nabla}U(\vec{x}). \tag{1.6}$$

So, it can be written as

$$\dot{\vec{p}} = \vec{F}, \tag{1.7}$$

or in the cases of constant mass,

$$\vec{F} = m\vec{a}, \tag{1.8}$$

that is nothing but Newton's second law.

Then, we've learned that, in fact, Euler-Lagrange equations for a given Lagrangian generalizes Newton's equations. Next, we'll introduce some useful notation.

1.3 Notation and conventions

We saw that mother tongue for classical mechanics is vectorial calculus. Physicist used to use indexes notation for vectors instead of a more geometrical notation, then a vector $\vec{A}$ is denoted by its components[1] A^i.

If we write (1.4) in this notation, we get

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i}\right) = \frac{\partial \mathcal{L}}{\partial x^i}. \tag{1.9}$$

Also,

$$\int_{t_1}^{t_2} dt \sum_{i=1}^{n} \left(\frac{\partial \mathcal{L}}{\partial x^i}\delta x^i - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}^i}\delta x^i\right) = \int_{t_1}^{t_2} dt \left(\frac{\partial \mathcal{L}}{\partial \vec{x}} \cdot \delta\vec{x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}} \cdot \delta\vec{x}\right). \tag{1.10}$$

Usually, we can drop the sum symbol by taking the sum convention (called Einstein's convention), that is, repeated indexes one upper and one lower implies sum over all possible values of the index. Then,

$$\int_{t_1}^{t_2} dt \left(\frac{\partial \mathcal{L}}{\partial x^i}\delta x^i - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}^i}\delta x^i\right) = \int_{t_1}^{t_2} dt \left(\frac{\partial \mathcal{L}}{\partial \vec{x}} \cdot \delta\vec{x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}} \cdot \delta\vec{x}\right). \tag{1.11}$$

1.4 Constrains and Degrees of freedom

Imagine that we're interested in the motion of a particle in a surface, say a sphere. It's well known that we can describe the motion by just two coordinates (θ, ϕ) because radius is constant. This is because sphere is a 2-dimensional restriction of a 3-dimensional space, therefore we need only 2-coordinates in order to describe the motion.

Then, for describing the motion of a particle (in general) we need a set of 6 parameters (3 positions and 3 velocities), but constrained the motion will lower the number of effective parameters needed. The degrees of freedom of a system is half of the number of effective parameters.

[1] Upper index means that it's a contra-variant component of the vector. There exist also co-variant components of a vector which are denoted by a lower index A_i. Contra-variant and co-variant components are related by the metric of the space where they are defined. In Euclidean space does not exist differences between them.

Chapter 2

Conserved Quantities

In this chapter we show how a global symmetry of the system gives rise to a conserved quantity, this is known as Nöther theorem. Then, we study a few but important applications of this theorem, where we found conditions for energy, momentum and angular momentum conservation.

2.1 Nöther theorem

2.2 Energy conservation

2.3 Momentum conservation

2.4 Angular Momentum conservation

Chapter 3

Integration of the equations of motion

Chapter 4

Canonical equations

Appendix A

Geometry: Conics

This chapter gain importance when one is interested in Kepler's problem, i.e., a particle in a potential $U(\vec{r}) \sim |\vec{r}|^{-1}$. Here we give a brief review of conics, further treatment can be found in a book on analytical geometry.

A.1 Circumference

A circumference is described by the equation[1]

$$x^2 + y^2 = R^2, \tag{A.1}$$

where R is the radius.

A.2 Ellipse

An ellipse is described by the equation,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{A.2}$$

with a and b the major and minor radii, respectively[2], i.e., $a > b$. Foci lie at the major diameter at a distance c, such that, $c^2 = a^2 - b^2$. eccentricity, ϵ, is defined by

$$\epsilon = \frac{c}{a}. \tag{A.3}$$

Obviously, for a circumference $\epsilon = 0$.

[1]Next we consider just figures centered at the origin of coordinates, without lost of generality.

[2]We've chosen the large side of the ellipse oriented in x-axis.

We can also characterize a point on the ellipse by the radii from the foci,

$$r_1 = a + \epsilon x, \quad r_2 = a - \epsilon x. \tag{A.4}$$

A.3 Hyperbola

A hyperbola is described by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{A.5}$$

where a is the distance from the origin to the vertice and b such that the hyperbola goes asymptotically as

$$y = \pm \frac{b}{a} x. \tag{A.6}$$

The foci are located at $c = \sqrt{a^2 + b^2}$, again the eccentricity is $\epsilon = \frac{c}{a}$ but now since $c > a \Rightarrow \epsilon > 1$. In term of the focal radii,

$$r_1 = a + \epsilon x, \quad r_2 = a - \epsilon x. \tag{A.7}$$

A.4 Parabola

A parabola is described by the equation

$$y^2 = 2px, \tag{A.8}$$

where focus is located at $p/2$ from the vertice. The focal radius is $r = x + p/2$ and eccentricity is $\epsilon = 1$.

A.5 Conics in polar coordinates

All the conics named before can be written in polar coordinates by the equation

$$\rho = \frac{p}{1 - \epsilon \cos \theta}. \tag{A.9}$$

Appendix B

Special Functions

Special functions play an important rôle when one wants to solve the equation of motion for certain systems, of course we don't pretend nor try to cover the subject, but showing a punctual useful formulae such as definitions, special values and so on.

B.1 Gamma Function

Integral forms for the gamma function are

$$\begin{aligned}
\Gamma(z) &= \int_0^\infty dt e^{-t} t^{z-1} && \text{(B.1)}\\
&= x^z \int_0^\infty dt e^{-xt} t^{z-1} && \text{(B.2)}\\
&= \int_0^\infty dt \ln(t) e^{-t} (t-z) e^{z-1} && \text{(B.3)}\\
&= \int_{-\infty}^\infty dt \exp(zt - e^t). && \text{(B.4)}
\end{aligned}$$

Also,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}, \quad \Gamma(n+1) = n\Gamma(n). \qquad \text{(B.5)}$$

B.2 Beta Function

Beta function can be defined in integral form by

$$\beta(x) = \int_0^1 dt \frac{t^{x-1}}{1+t} \tag{B.6}$$

$$= \int_0^\infty dt \frac{e^{-xt}}{1+e^{-t}}. \tag{B.7}$$

There exists the beta function for two parameters, defined by

$$B(x,y) = \int_0^1 dt t^{x-1}(1-t)^{y-1} \tag{B.8}$$

$$= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{B.9}$$

$$= B(y,x). \tag{B.10}$$

B.3 Bessel Functions

Bessel functions are solutions to the ordinary differential equation

$$\frac{d^2}{dx^2} Z_\nu(x) + \frac{1}{x}\frac{d}{dx} Z_\nu(x) + \left(1 - \frac{\nu^2}{x^2}\right) Z_\nu(x) = 0. \tag{B.11}$$

They have a lot of properties that we won't discuss here.

B.4 Associated Legendre Polynomials

Associated Legendre polynomials are solutions to the ordinary differential equation

$$(1-x^2)\frac{d}{dx} P_\nu^\mu(x) - 2x \frac{d}{dx} P_\nu^\mu(x) + \left(\nu(\nu+1) - \frac{\mu^2}{1-x^2}\right) P_\nu^\mu(x) = 0. \tag{B.12}$$

Legendre polynomials are the particular case $\mu = 0$, and they are defined by the relation

$$P_\nu(x) = \frac{1}{2^\nu \nu!} \frac{d^\nu}{dx^\nu}(x^2-1)^\nu. \tag{B.13}$$

Also,

$$P_\nu^\mu(x) = (-1)\mu(1-x^2)^{\mu/2} \frac{d^\mu}{dx^\mu} P_\nu(x). \tag{B.14}$$

B.5 Hermite Polynomials

Hermite polynomials are the solution to the ordinary differential equation

$$\frac{d^2}{dx^2}H_n(x) - 2x\frac{d}{dx}H_n(x) + 2nH_n(x) = 0. \tag{B.15}$$

Their Rodrigues' formula is

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}, \tag{B.16}$$

and they're a basis for the $L^2(x)$ satisfying

$$\int_{-\infty}^{\infty} dx e^{-x^2} H_n(x)H_m(x) = \begin{cases} 0; & m \neq n \\ \sqrt{\pi}2^n n!; & m = n \end{cases} \tag{B.17}$$

B.6 Laguerre polynomials

Laguerre polynomials are solution to the ordinary differential equation

$$x\frac{d^2}{dx^2}L_n^\alpha(x) + (\alpha + 1 - x)\frac{d}{dx}L_n^\alpha(x) + nL_n^\alpha(x) = 0. \tag{B.18}$$

Their Rodrigues' formula is

$$L_n^\alpha(x) = \frac{1}{n!}e^x x^{-\alpha}\frac{d^n}{dx^n}\left(e^{-x}x^\alpha\right) \tag{B.19}$$

$$= \sum_{m=0}^{n}(-1)^m \binom{n+\alpha}{n-\alpha}\frac{x^m}{m!} \tag{B.20}$$

They're also a basis for $L^2(x)$, and its orthogonality is given by

$$\int_0^\infty dx e^{-x}x^\alpha L_m^\alpha(x)L_n^\alpha(x) = \begin{cases} 0; & m \neq n \\ \binom{n+\alpha}{n}\Gamma(1+\alpha); & m = n \end{cases}. \tag{B.21}$$