Topological Properties of Networks: A Qualitative Reasoning Approach

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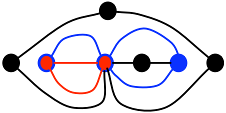
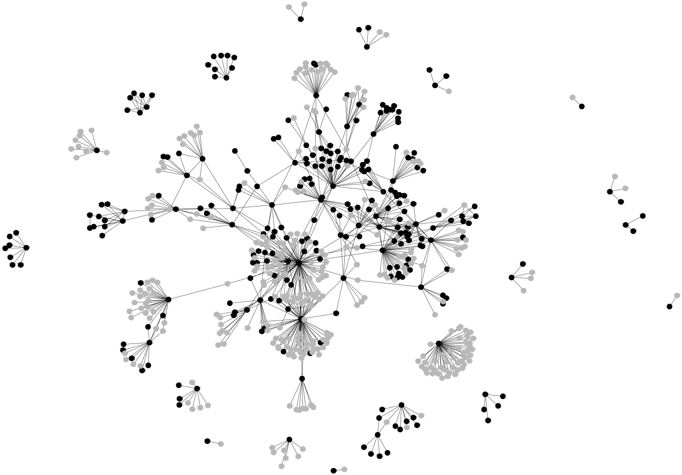
Motivation



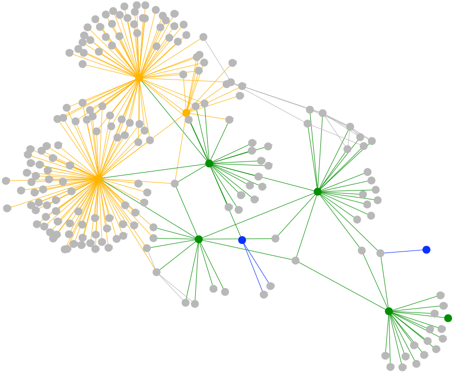
- Despite the advances on graph theory, little research has addressed formally relations –particularly topological relations– between networks that are part of a bigger network.

- Topological relations include binary relations that are invariant under continuous topological transformations (translation, rotation, and scaling).

Example: Chilean Research Community

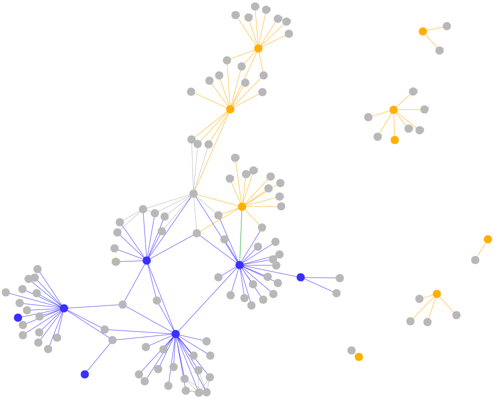


Example: Database - Web



- Blue: Database community
- Orange: Web community
- Green: Intersection of Database and Web community
- Gray: Co-authors outside these communities

Example: Database - Distributed Systems



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Motivation

- An important extension beyond the power of traditional query languages for graph and networks is the incorporation of topological properties into the primitives of query languages.
- Some typical queries are:
  Are the networks connected?
  Is a network completely included in the other one?
  Is there any element of a network that can connect two disconnected networks?
- A formalism for topological relations serves as a tool to identify and derive systematically them while avoiding redundant and contradictions.
- It also helps proving the completeness of the set of relationships and reasoning about them.
- Algorithms to determine relationships can be specified exactly, and mathematically sound models will help to define formally the relationships.

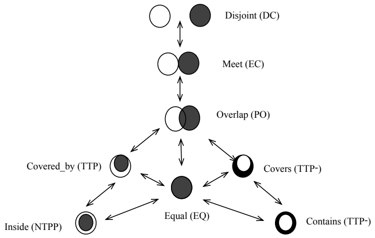
Approach

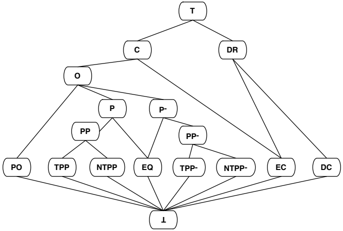
- Topological properties are related to the concept of connectivity, upon which different relations may be defined; for example, overlapping, inside, disjoint and meet.
- Topological relations have been extensively studied in the spatial domain, from where we borrowed some ideas.
- An abstract model based on algebraic properties of network servers as a formalism for defining the good objects and operations in the domain, and leads us conclude that natural objects in this domain are a generalization of the notion of graph.

Topological Qualitative Reasoning

- A qualitative reasoning describes situation in terms of variables whose values are drawn from domains with a small and predetermined number of possible values.
- In the context of topological qualitative reasoning, these variables usually represent topological relation.
- Fundamental concepts for topological qualitative reasoning are contact, parthood and boundary.
- Two well known models for topological relations in the spatial domain are: the Region Connected Calculus or RCC (Randell et al. 1992) and the point-set topological model or 9-Intsersection model (Egenhofer and Franzosa 1991). While point-set topological model has good computational properties, axiomatic theories as RCC are richer in their expressive power.

Topological Spatial Relations





RCC Model

- The RCC uses regions of space instead of points of classical geometry as primitives to define topological relations.
- It uses a primitive notion of connectivity defined as a binary predicate $C(x, y)$, whose semantics is that of "x is connected to y."
- Then definitions and axioms for topological relations are:

Relation	Interpretation	Definition
$DC(x, y)$	x is disconnected from y	$\neg C(x, y)$
$P(x, y)$	x is a part of y	$\forall z(C(z, x) \rightarrow C(z, y))$
$PP(x, y)$	x is a proper part of y	$P(x, y) \wedge \neg P(y, x)$
$EQ(x, y)$	x is equivalent with y	$P(x, y) \wedge P(y, x)$
$O(x, y)$	x overlaps y	$\exists z(P(z, x) \wedge P(z, y))$
$DR(x, y)$	x is discrete from y	$\neg O(x, y)$
$PO(x, y)$	x partially overlaps y	$O(x, y) \wedge \neg P(x, y) \wedge \neg P(y, x)$
$EC(x, y)$	x is externally connected to y	$C(x, y) \wedge \neg O(x, y)$
$TPP(x, y)$	x is a tangential proper part of y	$PP(x, y) \wedge \exists z(EC(z, x) \wedge EC(z, y))$
$NTPP(x, y)$	x is a nontangential proper part of y	$PP(x, y) \wedge \neg\exists z(EC(z, x) \wedge EC(z, y))$

Point-Set Topological Model

- Relations are defined by the emptiness or non-emptiness of the intersection between boundaries, interior and exterior of spatial objects.
- For topological relations between regions in 2D, where $^\circ$ is the interior, δ is the boundary and $^-$ is the exterior of a spatial objects, the following relations are defined :

DC	A°	δA	A^-
B°	0	0	1
δB	0	0	1
B^-	1	1	1

EC	A°	δA	A^-
B°	0	0	1
δB	0	1	1
B^-	1	1	1

PO	A°	δA	A^-
B°	1	1	1
δB	1	1	1
B^-	1	1	1

TPP	A°	δA	A^-
B°	1	0	0
δB	1	1	0
B^-	1	1	1

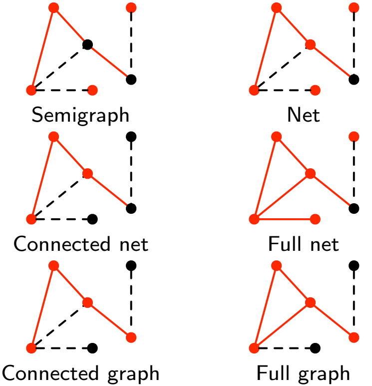
NTPP	A°	δA	A^-
B°	1	0	0
δB	1	0	0
B^-	1	1	1

EQ	A°	δA	A^-
B°	1	0	0
δB	0	1	0
B^-	0	0	1

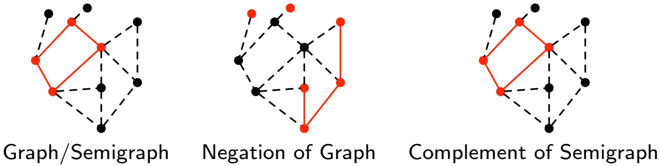
Basic Notions (1/3)

Definition: Let $U = (V_U, E_U)$ be a graph.

1. A semigraph over U is a pair (V, E) , where $V \subseteq V_U$ and $E \subseteq E_U$
2. A net is a semigraph (V, E) such that for each $uv \in E$ it holds that either $u \in V$ or $v \in V$.



Graph versus Semigraph



Basic Notions (2/2)

Notations: Let $V \subseteq V_U$ be a set of nodes, let $E \subseteq E_U$ be a set of edges, and let G be an arbitrary semigraph. We will denote by V_G its set of nodes and by E_G its set of edges.

1. Use uv to denote the undirected edge $\{u, v\}$.
2. A node v and an edge e are *incident* if $e = vw$ for some w. For two semigraphs x, y, $inc(x, y)$ is true iff there is $v \in x$ and $e \in y$ (or viceversa) which are incident.
3. $inc(V)$ is the set of edges $\{uv \in E_U : u \in V \vee v \in V\}$. Similarly, $inc(E)$, is the set of nodes $\{u, v \in V_U : uv \in E\}$
4. $sg(V)$ will denote the semigraph $(V, inc(V))$. Similarly, $sg(E)$ will denote the semigraph $(inc(E), E)$, and $sg(G)$ will denote the semigraph $(V_G \cup inc(E_G), E_G \cup inc(V_G))$.

Basic Notions (2/3)

Definition: [Basic operations on semigraphs] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be semigraphs.

1. The *union* of G_1 and G_2 (denoted $G_1 \cup G_2$) is the semigraph $(V_1 \cup V_2, E_1 \cup E_2)$.
2. The *intersection* of G_1 and G_2 (denoted $G_1 \cap G_2$) is the semigraph $(V_1 \cap V_2, E_1 \cap E_2)$.
3. The *difference* of G_1 and G_2 (denoted $G_1 - G_2$) is the semigraph $(V_1 - V_2, E_1 - E_2)$. In particular, the *complement* of G_2, denoted G_2^c, is the semigraph $U - G_2$.

Proposition: The set of *semigraphs* with the operations of union, intersection and complement, together with 0 defined as $(\emptyset, \emptyset)$ and $1 = (U, U \times U)$ is a Boolean algebra.

Algebraic Structure of Networks

- We can enrich the Boolean Algebra structure of semigraphs by defining a closure operator, and therefore, a structure of topological space.
- Closure operators must satisfy basic properties:

$$\begin{aligned} cl(\emptyset) &= \emptyset \\ G &\subseteq cl(G) \\ cl(cl(G)) &= cl(G) \\ cl(G \cup H) &= cl(G) \cup cl(H) \end{aligned}$$

- Based on the properties of closure, we can define closure over semigraphs in two ways:

$$\begin{aligned} cl_V(G) &= sg(V(G)) = (V(G), inc(V(G))) \\ cl_E(G) &= sg(E(G)) = (inc(E(G)), E(G)) \end{aligned}$$

Heyting Algebras

- Heyting algebra provides an elegant and natural theory for parthood, which have a closed connection to the notion of topological relations.
- **Definition:** A Heyting Algebra is a distributive lattice, A, equipped with a binary operation $a \Rightarrow b$, where $\Rightarrow$ satisfies

$$x \leq (a \Rightarrow b) \text{ iff } a \wedge x \leq b.$$

From here, one defines a *pseudo-complement* $\neg a$ as $a \Rightarrow 0$.
Dually, if there is a top element 1, and for every pair of elements a, b, the set of solutions of $b \leq a \vee x$ has a least element, we have a co-Heyting algebra. As in the case of Heyting algebras, the latter condition defines an operation $b \setminus a$, or in other words, the existence of a closed operation $\setminus$ which satisfies

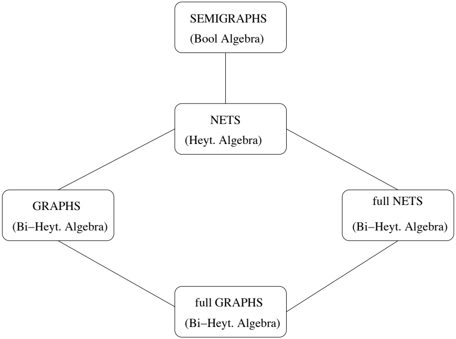
$$(b \setminus a) \leq x \text{ iff } b \leq a \vee x.$$

From here, one defines a *co-pseudo complemente* as $\sim a$ as $1 \sim a$.
If the order is a Heyting Algebra and a co-Heyting algebra, we have a *Bi-Heyting* algebra.
Finally, the order is a *Boolean Algebra* iff the pseudo-negation $\neg$ is really a negation, namely $\neg\neg x = x$.

Heyting Algebras via Closure Operators

- Every topology provides a complete Heyting algebra in the form of its open set lattice.
- A Heyting algebra for graphs: objects are open sets; operations are set-theoretical union and intersection; and the element $A \Rightarrow B$ is the interior of the union of $A^c \cup B$, where A^c denotes the complement of the open set A.
- A Heyting algebra for nets: objects are open sets; operations are the standard union, the intersection $G_1 \cup G_2$ is $(G_1 \cap G_2) \cap sg(V(G_1 \cap G_2))$, and $G_1 \Rightarrow G_2$ is $sg(V(G_1)^c) \cup G_2$.

Overall Algebraic Structure of Networks



Possible domains of objects to be considered, ordered by inclusion. Full Nets are nets which include all edges between their nodes, and full graphs are graphs with all edges between their nodes.

Heyting Algebra: Pointless Definitions

- We start by defining a Connectivity relation: $C(x, y)$ is true iff there is a path from x to y in $x \cup y$.
- Then, interesting relations can be derived from the Connectivity relation:

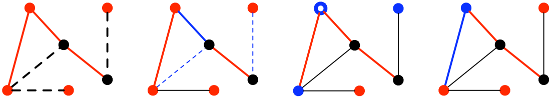
Relation	RCC	semigraph
$DC(x, y)$	$\neg C(x, y)$	No path between x and y in $x \cup y$
$P(x, y)$	$\forall z(C(z, x) \rightarrow C(z, y))$	$x \subseteq y$
$PP(x, y)$	$P(x, y) \wedge \neg P(y, x)$	$x \subset y$
$EQ(x, y)$	$P(x, y) \wedge P(y, x)$	$x = y$
$O(x, y)$	$\exists z(P(z, x) \wedge P(z, y))$	$x \cap y \neq \emptyset$
$DR(x, y)$	$\neg O(x, y)$	$x \cap y = \emptyset$
$PO(x, y)$	$O(x, y) \wedge \neg P(x, y) \wedge \neg P(y, x)$	$x \cap y \neq \wedge x \nsubseteq y \wedge y \nsubseteq x$
$EC(x, y)$	$C(x, y) \wedge \neg O(x, y)$	$x \cap y = \emptyset \wedge inc(x, y)$
$TPP(x, y)$	$PP(x, y) \wedge \exists z(EC(z, x) \wedge EC(z, y))$	$x \subset y \wedge inc(x, y^c)$
$NTPP(x, y)$	$PP(x, y) \wedge \neg\exists z(EC(z, x) \wedge EC(z, y))$	$x \subset y \wedge \neg inc(x, y^c)$

Point-set Definitions: Basic Notions

Let U be the universal graph, and H a semigraph in U.

- The boundary of a semigraph H (in U), denoted $\partial(H)$, is the set of edges which are incident to H and its complement, i.e., the set of edges uv of U such that $u \in H$ and $v \notin H$. (Note that edges $uv \notin H$ with $u \in H$ and $v \in H$ are not in the boundary).
  In particular, we define $\delta(H) = \partial(H) \cap H$ as the *real boundary*.
- The frontier of a semigraph H (in U), denoted $fr(H)$, is the set of nodes of H adjacent to nodes not in H. (Or equivalently: the set of nodes of H incident to $\partial(H)$.)
  In particular, we define $fr'(H)$, the *real* frontier, as the subset of the nodes of $fr(H)$ incident to edges not in H.
- The interior of a semigraph H (in U), denoted $interior(H)$, is the semigraph consisting of all nodes and edges of H not incident with elements not in H.
- The closure of a semigraph H (in U), denoted $cl(H)$, is the semigraph $H \cup \partial(H)$.

Basic Notions



Several useful properties of our definitions are:

1. $interior(H) = \emptyset$ if and only if $V(H) \subseteq fr(H)$.
2. $\partial(H) = \emptyset$ if and only if H is a maximal connected component of U.
3. $A \cap B = \emptyset$ if and only if $\partial(A)$ does not intersect B and $\partial(B)$ does not intersect A.
4. A and B are disconnected if and only if $A \cap B = \emptyset$ and $\partial(A) \cap \partial(B) = \emptyset$.
5. $\partial(A) \cap \partial(B) \neq \emptyset$ if and only if there is edge uv with $u \in A$ and $v \in B$ or $uv \in fr(A) \cap fr(B)$.
6. $\partial(A) \cap cl(B) = \emptyset$ implies either $A \subseteq B$ or $A \cap B = \emptyset$.

Intersection Model

	$interior(H_2) \cup fr'(H_2)$	$\partial(H_2)$
$interior(H_1) \cup fr'(H_1)$	0	0
$\partial(H_1)$	0	0

- Semigraphs are open so, semigraphs may not have boundary
- There are 2^4 possible 2x2 matrices
- Further refinement may allow us to distinguish interesting cases involving frontiers

Future Work

- We plan to refine some definitions to distinguish particular interesting cases of nets.
- We plan to relate the two approaches: pointless and point-set definitions of topological relations.
- We plan to study efficient algorithms for detecting the connectivity properties of networks based on these primitives.