Entanglement Entropy from Holography
Part 2

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Replica formula

Calabrese and Cardy, Journal of Physics A 42, 50

- Replica Trick to compute EE from Renyi entropies as the $n \to 1$ limit:

$$S_A^{(n)} = \frac{1}{1-n} \ln Tr\left(\rho_A^n\right)$$

$$S_A = -Tr \rho_A \ln \rho_A$$

$$S_A = \lim_{n \to 1} S_A^{(n)}$$

- In QM, partition function given by trace of density matrix over Hilbert space as

$$Z = Tr\left(\rho\right)$$

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Density matrix on thermal CFT

- In thermal CFT, density matrix given by Gibbs state. Represented with path integral as

$$\rho_{\phi_x,\phi'_x} = Z^{-1}(\beta)\left\langle \phi_x \middle| e^{-\beta H} \middle| \phi'_x \right\rangle = Z^{-1} \int_{\substack{\phi(x,0)=\phi_x \\ \phi(x,\beta)=\phi'_x}} [D\phi]\, e^{-S_E}$$

- Reduced density matrix (2D) by identifying fields outside region and tracing over configurations:

$$(\rho_A)_{\phi_x,\phi'_{x'}} = \int_{\substack{\phi_x=\phi'_x=\psi_x \\ x\in A^c}} [D\psi_x]\left(\rho_{\phi_x,\phi'_x}\right)$$

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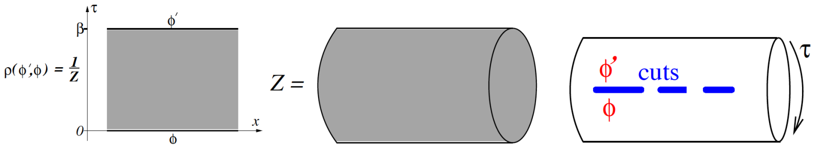

Figure 1. From density matrix to reduced density matrix. Left: Path integral representation of $\rho(\phi|\phi')$. Center: The partition function Z is obtained by sewing together the edges along $\tau = 0$ and $\tau = \beta$ to form a cylinder of circumference β. Right: The reduced density matrix ρ_A is obtained by sewing together only those points x which are not in A.

Partition function for replica CFT

- Definition of replica partition function to compute trace of power of reduced density matrix

$$Tr\left(\rho_A^n\right) = \frac{Z_n\left(A\right)}{Z^n}$$

- Replica CFT made from n copies of original CFT glued at the interval. Multi-sheeted Riemann surface:

$$Z_n\left(A\right) = Z^{-1}\int \left[D\phi\right]_{M_n} \exp\left[-\int_{M_n} dxd\tau\mathcal{L}\left[\phi\right]\right]$$

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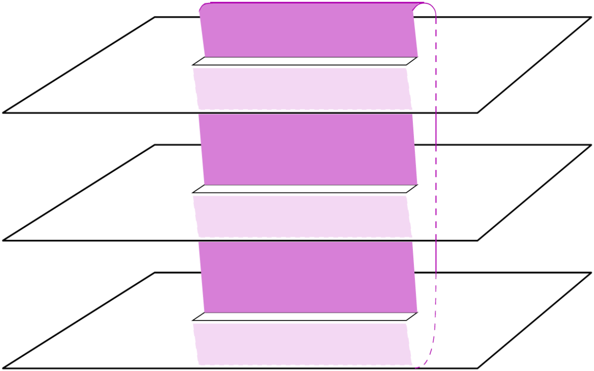

Figure 2. A representation of the Riemann surface $\mathcal{R}_{3,1}$.

Partition function for replica CFT

- Partition function in terms of sum of Lagrangians on single copy of complex plane as

$$Z_n(A) = Z^{-1} \int_{R_{u,v}} [D\phi_1 \cdots D\phi_n] \times \exp\left[-\int_C dx d\tau \left(\mathcal{L}[\phi_1] + \ldots + \mathcal{L}[\phi_n] \right) \right]$$

- Restricted path integral:

$$\phi_i(x, 0^+) = \phi_{i+1}(x, 0^-)$$
$$x \in [u, v] \;;\; i = 1, \ldots, n \;;\; \phi_{i+n} = \phi_i$$

- Definition of replica Lagrangian as:

$$\mathcal{L}^{(n)}[\phi_1, \ldots, \phi_n] = \mathcal{L}[\phi_1] + \ldots + \mathcal{L}[\phi_n]$$

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Twist operators

- Introduce (local) Twist operators. They implement replica symmetry:

$$\mathcal{T}_n \ : \ i \to i+1$$
$$\widetilde{\mathcal{T}}_n : \ i+1 \to i$$
$$\mathcal{T}_n\left[\mathcal{L}^{(n)}\right] = \widetilde{\mathcal{T}}_n\left[\mathcal{L}^{(n)}\right] = \mathcal{L}^{(n)}$$

- Replica partition function as correlator of Twist operators:

$$Z_n(A) \propto \left\langle \mathcal{T}_n(u,0)\,\widetilde{\mathcal{T}}_n(v,0)\right\rangle_{\mathcal{L}^{(n)},C}$$

- VEVs of operators in replica CFT computed as

$$\langle O_i \rangle = \frac{\left\langle \mathcal{T}_n(u,0)\,\widetilde{\mathcal{T}}_n(v,0)\,O_i\right\rangle_{\mathcal{L}^{(n)},C}}{\left\langle \mathcal{T}_n(u,0)\,\widetilde{\mathcal{T}}_n(v,0)\right\rangle_{\mathcal{L}^{(n)},C}}$$

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Two-point functions and Ward identities for twist operators

- Consider VEV of E-M tensor over multi-sheeted surface:

$$\left\langle T^{(n)}(z)\right\rangle_{M_n} = \frac{\left\langle \mathcal{T}_n(u,0)\,\widetilde{\mathcal{T}}_n(v,0)\,T^{(n)}(z)\right\rangle_{\mathcal{L}^{(n)},C}}{\left\langle \mathcal{T}_n(u,0)\,\widetilde{\mathcal{T}}_n(v,0)\right\rangle_{\mathcal{L}^{(n)},C}}$$

- Correlator constrained to the form

$$\left\langle \mathcal{T}_n(u,0)\,\widetilde{\mathcal{T}}_n(v,0)\right\rangle_{\mathcal{L}^{(n)},C} = \frac{C_{12}}{|u-v|^{2d_n}}$$

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- Twist operators as quasi-primaries, Ward identity given by

$$X \overset{\text{def.}}{=} \mathcal{T}_n(u,0)\,\widetilde{\mathcal{T}}_n(v,0)$$

$$\left\langle X T^{(n)}(z)\right\rangle_{\mathcal{L}^{(n)},C} = \left(\begin{array}{c} \frac{1}{z-u}\frac{\partial}{\partial u} + \frac{d_n}{(z-u)^2} \\ +\frac{1}{z-v}\frac{\partial}{\partial v} + \frac{d_n}{(z-v)^2} \end{array}\right)\langle X\rangle_{\mathcal{L}^{(n)},C}$$

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Uniformizing the interval

- Uniformizing transformation maps multi-sheeted surface with branch cut to single copy of complex plane:

$$z' = (\zeta)^{1/n} = \left(\frac{z-u}{z-v}\right)^{1/n}$$

- E-M tensor transforms as

$$T(z) = \left(\frac{dz'}{dz}\right)^2 T(z') + \frac{c}{12}\{z', z\}$$

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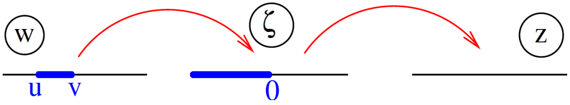

Figure 3. Uniformizing transformation for $\mathcal{R}_{n,1}$. $w \to \zeta = (w-u)/(w-v)$ maps the branch points to $(0, \infty)$. This is uniformized by the mapping $\zeta \to z = \zeta^{1/n}$.

Finding the conformal dimension of the twist operators

- Considering Schwarzian derivative, VEV of vacuum E-M tensor in multi-sheeted surface given by

$$\left\langle T^{(n)}(z) \right\rangle_{M_n} = n\frac{c}{12}\left\{z', z\right\} = \frac{c\left(n^2-1\right)}{24n}\frac{(v-u)^2}{(z-u)^2(z-v)^2}$$

- Comparing with previous form, conformal dimension is obtained as

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EE and ERE from conformal symmetry

- Trace of density matrix for replica CFT (partition function) proportional to

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$$S_A^{(n)} = \frac{1}{1-n}\ln Tr\left(\rho_A^n\right)$$

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Introductory comments about AdS spacetime

- Poincaré patch of AdS spacetime given by

$$ds_G^2 = \ell^2 \left(\frac{dz^2 + dx^\mu dx_\mu}{z^2} \right),$$

with $z \geq 0$.

- Isometry group of AdS is isomorphic to the conformal group at the Minkowski boundary. Dilatation symmetry explicit as $x^\mu \to \lambda x^\mu$ and $z \to \lambda z$.
- Other useful parametrizations given by

$$ds_G^2 = \frac{\ell^2 d\rho^2}{4\rho^2} + \left(\frac{dx^\mu dx_\mu}{\rho} \right). \tag{1}$$

- AAdS spacetimes defined as $R^{\mu\nu}_{\rho\sigma} \to -\frac{1}{\ell^2}\delta^{[\mu\nu]}_{[\rho\sigma]}$ at the conf. boundary. Defines divergent Weyl factor sources a conformal structure. Equivalence class of asymptotic metrics $[g_{(0)}]$.

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- Quantum field theory with conformal symmetry (global symmetry).
- Fileds and correlation functions are covariant under dilatations with a certain conformal weight.
- Form of 2-point and 3-point functions constrained by symmetry.
- Dimensionless coupling constants, no preferred scale.
- Conformal symmetry may be broken at the quantum level. Anomaly proportional to trace of energy-momentum tensor. Example: SU(N) gauge theory in even dimensions.
- In large N limit, planar diagrams of SU(N) are dominant. For strongly coupled regime, diagrams resemble Riemann surfaces. Genus expansion similar to string theory.

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- In the $N \to \infty$, $g_s \to 0$, $\alpha' \to 0$, $\lambda = 4\pi N g_s \gg 1$ regime, it is a duality between weakly coupled (super)gravity and strongly coupled (S)CFT at the conformal boundary.
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- In the $N \to \infty$, $g_s \to 0$, $\alpha' \to 0$, $\lambda = 4\pi N g_s \gg 1$ regime, it is a duality between weakly coupled (super)gravity and strongly coupled (S)CFT at the conformal boundary.
- Usually, duality is assumed in more general contexts and the phenomenology is studied (bottom-up approach).

Statement of the duality

- The partition function for the bulk AdS (super)gravity theory is equal to the generator of connected correlation functions of the (S)CFT.

$$Z_{AdS}\left[\phi_i\right] = Z_{CFT}\left[\phi_{0,i}\right] = e^{W\left[\phi_{0,i}\right]}$$

- Bulk fields at the boundary are mapped into external sources for the CFT operators. Quantum numbers of the global symmetry are matched on both sides. For example, boundary metric is source of energy-momentum tensor.
- In saddle-point approximation:

$$e^{-I_{AdS}\left[\phi_i\right]} \approx \left\langle e^{\int \phi_{0,i} O^i} \right\rangle_{CFT} = e^{W\left[\phi_{0,i}\right]}$$

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Holography and Renormalization

- Gravity computation to obtain CFT properties.
- AdS metric near the boundary admits FG expansion:

$$ds_G^2 = G_{\mu\nu}dx^\mu dx^\nu = \frac{\ell^2 d\rho^2}{4\rho^2} + \frac{1}{\rho}\left(g_{ij}^{(0)}(x) + \rho g_{ij}^{(1)}(x) + \dots\right)dx^i dx^j$$

- Energy-momentum tensor of CFT can be analyzed from gravity side. (Henningson and Skenderis [hep-th/9806086]; Imbimbo, Schwimmer, Theisen and Yankielowicz [hep-th/9910267]).
- Radial diff. in the bulk corresponds to Weyl transf. in the CFT. Anomaly can be computed.

$$\delta_\sigma g_{(0)}^{ij} = 2\sigma g_{(0)}^{ij}$$

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- Problem: AdS gravity action is divergent near boundary. Solution: Holographic renormalization. (Emparan, Johnson and Myers [hep-th/9903238]; Balasubramanian and Kraus [hep-th/9902121]; de Haro, Skenderis and Solodukhin [hep-th/0002230].)
- For Einstein-AdS action in the bulk, add GHY boundary term (well posed variational principle). Then add intrinsic counterterms covariant w.r.t. induced boundary metric to cancel divergences.
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Counterterm method

- Renormalized AdS gravity action: (Holographic Renormalization)

$$I_{ren} = \frac{1}{16\pi G}\int_M d^{d+1}x\sqrt{-G}\,(R-2\Lambda) - \frac{1}{8\pi G}\int_{\partial M} d^d x\,\sqrt{-h}\,K + \int_{\partial M} d^d x\,\mathcal{L}_{ct}(h,\mathcal{R},\nabla\mathcal{R})$$

$$\Lambda = -\frac{d\,(d-1)}{2\ell^2}$$

- Renormalized quasi-local stress tensor: $T^{ij}_{ren}[h] = \frac{2}{\sqrt{-h}}\frac{\delta I_{ren}}{\delta h_{ij}}$.
- Background-independent charges: Vacuum energy for global AdS (Casimir energy for boundary CFT) [Balasubramanian and Kraus [hep-th/9902121]]

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Counterterm Series (Einstein-AdS)

- The counterterm Lagrangian for Einstein-AdS is given by

$$
\begin{aligned}
\mathcal{L}_{ct} =& \frac{d-1}{\ell}\sqrt{-h} + \frac{\ell\sqrt{-h}}{2(d-2)}\mathcal{R} \\
&+ \frac{\ell^3\sqrt{-h}}{2(d-2)^2(d-4)}\left(\mathcal{R}^{ij}\mathcal{R}_{ij} - \frac{d}{4(d-1)}\mathcal{R}^2\right) \\
&+ \frac{\ell^5\sqrt{-h}}{(d-2)^3(d-4)(d-6)}\left(\frac{3d-2}{4(d-1)}\mathcal{R}\mathcal{R}^{ij}\mathcal{R}_{ij}\right. \\
&- \frac{d(d+2)}{16(d-1)^2}\mathcal{R}^3 - 2\mathcal{R}^{ij}\mathcal{R}^{kl}\mathcal{R}_{ijkl} \\
&- \left.\frac{d}{4(d-1)}\nabla_i\mathcal{R}\nabla^i\mathcal{R} + \nabla^k\mathcal{R}^{ij}\nabla_k\mathcal{R}_{ij}\right) + \ldots
\end{aligned}
$$

Holographic Renormalization

- FG expansion of boundary metric in powers of ρ given by

$$g_{ij}(x,\rho) = g_{(0)ij}(x) + \rho g_{(1)ij}(x) + \rho^2 g_{(2)ij}(x) + \cdots .$$

- $g_{(0)ij}$ is the boundary data for the *holographic reconstruction of the spacetime*, i.e., solving $g_{(k)}$ as a covariant funtional of $g_{(0)}$
- $g_{(k)}$ coefficients obtained from $g_{(0)}$ by using Einstein's equations.
- FG form preserved under radial diffeomorphisms. Correspond to Weyl transf. of $g_{(0)}$.

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- For instance

$$g_{(1)ij} = \frac{1}{d-2}\left(\mathcal{R}_{(0)ij} - \frac{1}{2(d-1)} g_{(0)ij}\mathcal{R}_{(0)}\right)$$

$$\begin{aligned} g_{(2)ij} = \frac{1}{d-4}\Big(& -\frac{1}{8(d-1)}\nabla_i\nabla_j\mathcal{R}_{(0)} + \frac{1}{4(d-2)}\nabla_k\nabla^k\mathcal{R}_{(0)ij} \\ & -\frac{1}{8(d-1)(d-2)} g_{(0)ij}\nabla_k\nabla^k\mathcal{R}_{(0)} - \frac{1}{2(d-2)}\mathcal{R}^{kl}_{(0)}\mathcal{R}_{(0)ikjl} \\ & +\frac{(d-4)}{2(d-2)^2}\mathcal{R}_{(0)ik}\mathcal{R}^{k}_{(0)j} + \frac{1}{(d-1)(d-2)^2}\mathcal{R}_{(0)}\mathcal{R}_{(0)ij} \\ & +\frac{1}{4(d-2)^2}\mathcal{R}_{(0)kl}\mathcal{R}^{kl}_{(0)} g_{(0)ij} - \frac{3d}{16(d-1)^2(d-2)^2}\mathcal{R}^2_{(0)} g_{(0)ij}\Big) \end{aligned}$$

- Holographic stress tensor: $T^{ij}[g_{(0)}, g_{(d/2)}] = \lim_{\rho\to 0}\left(\frac{1}{\rho^{d/2-1}} T^{ij}[h]\right) = \langle T_{ij}\rangle_{CFT}$. Contains the holographic information of the theory (e.g., Weyl anomaly)

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Holographic Renormalization Recap

- Using a suitably renormalized gravity action, the EE will give a finite result.
- Holographic Renormalization counterterms series determined by cancellation of divergencies.
- Number of CTs grows with the dimensionality of spacetime.
- No closed-form expression for the counterterm series in general dimensions. Different expressions for different theories of gravity.

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