# Entanglement Entropy from Holography
# Part 1

Ignacio J. Araya
araya.quezada.ignacio@gmail.com
Universidad Andrés Bello

Universidad de Concepción - Concepción - Chile

August 22, 2019

# Entanglement entropy

- Measures amount of entanglement of subsystem $A$ with rest of the system $(A^c)$.
- Assume the total Hilbert space factorizes into the Hilbert spaces of the subsystems $A$ and $A^c$

$$\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$$

- EE defined as the von Neumann Entropy of reduced density matrix for subsystem $A$:

$$S_{EE} = -tr\left(\widehat{\rho}_A \ln \widehat{\rho}_A\right).$$

- Density matrix of subsystem $A$ is defined as

$$\rho_A = tr_{A^c}\left(\rho_{tot}\right) = \langle i_{A^c} | \, \rho_{tot} \, | i_{A^c} \rangle$$

# Entanglement entropy

- Measures amount of entanglement of subsystem $A$ with rest of the system ($A^c$).
- Assume the total Hilbert space factorizes into the Hilbert spaces of the subsystems $A$ and $A^c$

$$\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$$

- EE defined as the von Neumann Entropy of reduced density matrix for subsystem $A$:

$$S_{EE} = -tr\left(\widehat{\rho}_A \ln \widehat{\rho}_A\right).$$

- Density matrix of subsystem $A$ is defined as

$$\rho_A = tr_{A^c}\left(\rho_{tot}\right) = \langle i_{A^c} | \, \rho_{tot} \, | i_{A^c} \rangle$$

# Entanglement entropy

- Measures amount of entanglement of subsystem $A$ with rest of the system ($A^c$).
- Assume the total Hilbert space factorizes into the Hilbert spaces of the subsystems $A$ and $A^c$

$$\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$$

- EE defined as the von Neumann Entropy of reduced density matrix for subsystem $A$:

$$S_{EE} = -tr\left(\widehat{\rho}_A \ln \widehat{\rho}_A\right).$$

- Density matrix of subsystem $A$ is defined as

$$\rho_A = tr_{A^c}\left(\rho_{tot}\right) = \langle i_{A^c} | \rho_{tot} | i_{A^c} \rangle$$

# Entanglement entropy

- Measures amount of entanglement of subsystem $A$ with rest of the system ($A^c$).
- Assume the total Hilbert space factorizes into the Hilbert spaces of the subsystems $A$ and $A^c$

$$\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$$

- EE defined as the von Neumann Entropy of reduced density matrix for subsystem $A$:

$$S_{EE} = -tr\left(\widehat{\rho}_A \ln \widehat{\rho}_A\right).$$

- Density matrix of subsystem $A$ is defined as

$$\rho_A = tr_{A^c}\left(\rho_{tot}\right) = \langle i_{A^c}|\, \rho_{tot}\, |i_{A^c}\rangle$$

# Entanglement Entropy

- $\{|i_{A^c}\rangle\}_i$ is a basis of $\mathcal{H}_{A^c}$
- Reduced density matrix given by $\widehat{\rho}_A = \frac{\rho_A}{tr(\rho_A)}$, defined with total probability equal to 1.
- For the case of QFTs, $\mathcal{H}_{tot}$ is infinite dimensional and a tensor product of Hilbert subspaces at each spatial point. $A$ is a spatial region.
- For holographic CFTs, with AAdS gravity duals, EE can be obtained using AdS/CFT tools from a geometric computation in the bulk.

# Entanglement Entropy

- $\{|i_{A^c}\rangle\}_i$ is a basis of $\mathcal{H}_{A^c}$
- Reduced density matrix given by $\widehat{\rho}_A = \frac{\rho_A}{tr(\rho_A)}$, defined with total probability equal to 1.
- For the case of QFTs, $\mathcal{H}_{tot}$ is infinite dimensional and a tensor product of Hilbert subspaces at each spatial point. $A$ is a spatial region.
- For holographic CFTs, with AAdS gravity duals, EE can be obtained using AdS/CFT tools from a geometric computation in the bulk.

# Entanglement Entropy

- $\{|i_{A^c}\rangle\}_i$ is a basis of $\mathcal{H}_{A^c}$
- Reduced density matrix given by $\widehat{\rho}_A = \frac{\rho_A}{tr(\rho_A)}$, defined with total probability equal to 1.
- For the case of QFTs, $\mathcal{H}_{tot}$ is infinite dimensional and a tensor product of Hilbert subspaces at each spatial point. $A$ is a spatial region.
- For holographic CFTs, with AAdS gravity duals, EE can be obtained using AdS/CFT tools from a geometric computation in the bulk.

# Entanglement Entropy

- $\{|i_{A^c}\rangle\}_i$ is a basis of $\mathcal{H}_{A^c}$
- Reduced density matrix given by $\widehat{\rho}_A = \frac{\rho_A}{tr(\rho_A)}$, defined with total probability equal to 1.
- For the case of QFTs, $\mathcal{H}_{tot}$ is infinite dimensional and a tensor product of Hilbert subspaces at each spatial point. $A$ is a spatial region.
- For holographic CFTs, with AAdS gravity duals, EE can be obtained using AdS/CFT tools from a geometric computation in the bulk.

# Renyi Entropy

- Knowledge of all the Rényi entropies allows for reconstruction of the entanglement spectrum (the eigenvalues of $\widehat{\rho}_A$).
- m-th Rényi entropy defined as

$$S_m = -\frac{1}{m-1} \ln \left( tr \left( \widehat{\rho}_A^m \right) \right).$$

- In general, it is easier to compute than the EE.
- Analytic continuation of m-th Rényi entropy can be used to compute EE:

$$S_{EE} = \lim_{m \to 1} -\frac{1}{m-1} \ln \left( tr \left( \widehat{\rho}_A^m \right) \right).$$

# Renyi Entropy

- Knowledge of all the Rényi entropies allows for reconstruction of the entanglement spectrum (the eigenvalues of $\widehat{\rho}_A$).
- m-th Rényi entropy defined as

$$S_m = -\frac{1}{m-1} \ln \left( tr \left( \widehat{\rho}_A^m \right) \right).$$

- In general, it is easier to compute than the EE.
- Analytic continuation of m-th Rényi entropy can be used to compute EE:

$$S_{EE} = \lim_{m \to 1} -\frac{1}{m-1} \ln \left( tr \left( \widehat{\rho}_A^m \right) \right).$$

# Renyi Entropy

- Knowledge of all the Rényi entropies allows for reconstruction of the entanglement spectrum (the eigenvalues of $\widehat{\rho}_A$).
- m-th Rényi entropy defined as

$$S_m = -\frac{1}{m-1} \ln \left( tr \left( \widehat{\rho}_A^m \right) \right).$$

- In general, it is easier to compute than the EE.
- Analytic continuation of m-th Rényi entropy can be used to compute EE:

$$S_{EE} = \lim_{m \to 1} -\frac{1}{m-1} \ln \left( tr \left( \widehat{\rho}_A^m \right) \right).$$

# Renyi Entropy

- Knowledge of all the Rényi entropies allows for reconstruction of the entanglement spectrum (the eigenvalues of $\widehat{\rho}_A$).
- m-th Rényi entropy defined as

$$S_m = -\frac{1}{m-1} \ln\left(tr\left(\widehat{\rho}_A^m\right)\right).$$

- In general, it is easier to compute than the EE.
- Analytic continuation of m-th Rényi entropy can be used to compute EE:

$$S_{EE} = \lim_{m\to 1} -\frac{1}{m-1} \ln\left(tr\left(\widehat{\rho}_A^m\right)\right).$$

# Example, EE and EREs for a two spin system

- Consider two spin system in pure singlet state given by

$$|\Psi\rangle = \frac{1}{\sqrt{2}} |\uparrow\downarrow\rangle - \frac{1}{\sqrt{2}} |\downarrow\uparrow\rangle$$

- Intuitively, system exhibits entanglement. Measurement of left spin determines right spin orientation.
- Density matrix of system given by:

$$\rho = |\Psi\rangle \langle\Psi| = \frac{1}{2} |\uparrow\downarrow\rangle \langle\uparrow\downarrow| - \frac{1}{2} |\uparrow\downarrow\rangle \langle\downarrow\uparrow| - \frac{1}{2} |\downarrow\uparrow\rangle \langle\uparrow\downarrow| + \frac{1}{2} |\downarrow\uparrow\rangle \langle\downarrow\uparrow|$$

- The reduced density matrix for the left spin is given by

$$\widehat{\rho}_L = tr_{\mathcal{H}_R}(\rho) = \sum_{i\in\{\downarrow,\uparrow\}} \langle i_R| \rho |i_R\rangle = \frac{1}{2} |\downarrow_L\rangle \langle\downarrow_L| + \frac{1}{2} |\uparrow_L\rangle \langle\uparrow_L|$$

- Reduced density matrix describes configuration that is not pure state.

# Example, EE and EREs for a two spin system

- Consider two spin system in pure singlet state given by

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle$$

- Intuitively, system exhibits entanglement. Measurement of left spin determines right spin orientation.
- Density matrix of system given by:

$$\rho = |\Psi\rangle\langle\Psi| = \frac{1}{2}|\uparrow\downarrow\rangle\langle\uparrow\downarrow| - \frac{1}{2}|\uparrow\downarrow\rangle\langle\downarrow\uparrow| - \frac{1}{2}|\downarrow\uparrow\rangle\langle\uparrow\downarrow| + \frac{1}{2}|\downarrow\uparrow\rangle\langle\downarrow\uparrow|$$

- The reduced density matrix for the left spin is given by

$$\widehat{\rho}_L = tr_{\mathcal{H}_R}(\rho) = \sum_{i\in\{\downarrow,\uparrow\}} \langle i_R|\rho|i_R\rangle = \frac{1}{2}|\downarrow_L\rangle\langle\downarrow_L| + \frac{1}{2}|\uparrow_L\rangle\langle\uparrow_L|$$

- Reduced density matrix describes configuration that is not pure state.

# Example, EE and EREs for a two spin system

- Consider two spin system in pure singlet state given by

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle$$

- Intuitively, system exhibits entanglement. Measurement of left spin determines right spin orientation.
- Density matrix of system given by:

$$\rho = |\Psi\rangle\langle\Psi| = \frac{1}{2}|\uparrow\downarrow\rangle\langle\uparrow\downarrow| - \frac{1}{2}|\uparrow\downarrow\rangle\langle\downarrow\uparrow| - \frac{1}{2}|\downarrow\uparrow\rangle\langle\uparrow\downarrow| + \frac{1}{2}|\downarrow\uparrow\rangle\langle\downarrow\uparrow|$$

- The reduced density matrix for the left spin is given by

$$\widehat{\rho}_L = tr_{\mathcal{H}_R}(\rho) = \sum_{i\in\{\downarrow,\uparrow\}} \langle i_R|\rho|i_R\rangle = \frac{1}{2}|\downarrow_L\rangle\langle\downarrow_L| + \frac{1}{2}|\uparrow_L\rangle\langle\uparrow_L|$$

- Reduced density matrix describes configuration that is not pure state.

# Example, EE and EREs for a two spin system

- Consider two spin system in pure singlet state given by

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle$$

- Intuitively, system exhibits entanglement. Measurement of left spin determines right spin orientation.
- Density matrix of system given by:

$$\rho = |\Psi\rangle\langle\Psi| = \frac{1}{2}|\uparrow\downarrow\rangle\langle\uparrow\downarrow| - \frac{1}{2}|\uparrow\downarrow\rangle\langle\downarrow\uparrow| - \frac{1}{2}|\downarrow\uparrow\rangle\langle\uparrow\downarrow| + \frac{1}{2}|\downarrow\uparrow\rangle\langle\downarrow\uparrow|$$

- The reduced density matrix for the left spin is given by

$$\widehat{\rho}_L = tr_{\mathcal{H}_R}(\rho) = \sum_{i\in\{\downarrow,\uparrow\}} \langle i_R|\rho|i_R\rangle = \frac{1}{2}|\downarrow_L\rangle\langle\downarrow_L| + \frac{1}{2}|\uparrow_L\rangle\langle\uparrow_L|$$

- Reduced density matrix describes configuration that is not pure state.

# Example, EE and EREs for a two spin system

- Consider two spin system in pure singlet state given by

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left|\uparrow\downarrow\right\rangle - \frac{1}{\sqrt{2}}\left|\downarrow\uparrow\right\rangle$$

- Intuitively, system exhibits entanglement. Measurement of left spin determines right spin orientation.
- Density matrix of system given by:

$$\rho = |\Psi\rangle\langle\Psi| = \frac{1}{2}\left|\uparrow\downarrow\right\rangle\left\langle\uparrow\downarrow\right| - \frac{1}{2}\left|\uparrow\downarrow\right\rangle\left\langle\downarrow\uparrow\right| - \frac{1}{2}\left|\downarrow\uparrow\right\rangle\left\langle\uparrow\downarrow\right| + \frac{1}{2}\left|\downarrow\uparrow\right\rangle\left\langle\downarrow\uparrow\right|$$

- The reduced density matrix for the left spin is given by

$$\widehat{\rho}_L = tr_{\mathcal{H}_R}(\rho) = \sum_{i\in\{\downarrow,\uparrow\}} \langle i_R|\,\rho\,|i_R\rangle = \frac{1}{2}\left|\downarrow_L\right\rangle\left\langle\downarrow_L\right| + \frac{1}{2}\left|\uparrow_L\right\rangle\left\langle\uparrow_L\right|$$

- Reduced density matrix describes configuration that is not pure state.

# Example, EE and EREs for a two spin system

- EE given by

$$S_{EE} = -tr\left(\widehat{\rho}_L \ln \widehat{\rho}_L\right)$$

$$S_{EE} = -tr\left(\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \ln \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\right) = \ln 2$$

- m-th Rényi entropy given by

$$S_m = -\frac{1}{m-1} \ln\left(tr\left(\widehat{\rho}_L^m\right)\right)$$

$$S_m = -\frac{1}{m-1} \ln\left(tr\left(\begin{bmatrix} \frac{1}{2^m} & 0 \\ 0 & \frac{1}{2^m} \end{bmatrix}\right)\right) = \ln 2$$

- In this case, ERE is independent of m. Limit of m $\to$ 1 is trivial. Identity relating ERE and EE is satisfied.

# Example, EE and EREs for a two spin system

- EE given by

$$S_{EE} = -tr\left(\widehat{\rho}_L \ln \widehat{\rho}_L\right)$$

$$S_{EE} = -tr\left(\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \ln \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\right) = \ln 2$$

- m-th Rényi entropy given by

$$S_m = -\frac{1}{m-1} \ln\left(tr\left(\widehat{\rho}_L^m\right)\right)$$

$$S_m = -\frac{1}{m-1} \ln\left(tr\left(\begin{bmatrix} \frac{1}{2^m} & 0 \\ 0 & \frac{1}{2^m} \end{bmatrix}\right)\right) = \ln 2$$

- In this case, ERE is independent of m. Limit of m $\rightarrow$ 1 is trivial. Identity relating ERE and EE is satisfied.

# Example, EE and EREs for a two spin system

- EE given by

$$S_{EE} = -tr\left(\widehat{\rho}_L \ln \widehat{\rho}_L\right)$$

$$S_{EE} = -tr\left(\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \ln \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\right) = \ln 2$$

- m-th Rényi entropy given by

$$S_m = -\frac{1}{m-1} \ln\left(tr\left(\widehat{\rho}_L^m\right)\right)$$

$$S_m = -\frac{1}{m-1} \ln\left(tr\left(\begin{bmatrix} \frac{1}{2^m} & 0 \\ 0 & \frac{1}{2^m} \end{bmatrix}\right)\right) = \ln 2$$

- In this case, ERE is independent of m. Limit of m $\rightarrow$ 1 is trivial. Identity relating ERE and EE is satisfied.

# QFT: Noether Theorem

Di Francesco, Mathieu and Senechal, *Conformal Field Theory*

- Consider an arbitrary action in field theory:

$$S[\Phi] = \int d^d x \mathcal{L}(\Phi, \partial_\mu \Phi)$$

- Consider a transformation that acts on the coordinates and fields:

$$\Phi'(x') = \mathcal{F}(\Phi(x))$$

- Infinitesimal version given by

$$x'^\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a}$$

$$\Phi'(x') = \Phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x)$$

# QFT: Noether Theorem

Di Francesco, Mathieu and Senechal, *Conformal Field Theory*

- Consider an arbitrary action in field theory:

$$S[\Phi] = \int d^d x \mathcal{L}(\Phi, \partial_\mu \Phi)$$

- Consider a transformation that acts on the coordinates and fields:

$$\Phi'(x') = \mathcal{F}(\Phi(x))$$

- Infinitesimal version given by

$$x'^\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a}$$

$$\Phi'(x') = \Phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x)$$

# QFT: Noether Theorem

Di Francesco, Mathieu and Senechal, *Conformal Field Theory*

- Consider an arbitrary action in field theory:

$$S\left[\Phi\right] = \int d^d x \mathcal{L}\left(\Phi, \partial_\mu \Phi\right)$$

- Consider a transformation that acts on the coordinates and fields:

$$\Phi'\left(x'\right) = \mathcal{F}\left(\Phi\left(x\right)\right)$$

- Infinitesimal version given by

$$x'^\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a}$$

$$\Phi'\left(x'\right) = \Phi\left(x\right) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}\left(x\right)$$

# QFT: Noether Theorem

- The Jacobian and Jacobian matrix of the coordinate transformation is given by

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \delta^{\mu}_{\nu} + \partial_{\nu}\left(\omega_a \frac{\delta x^{\mu}}{\delta \omega_a}\right)$$
$$\frac{\partial x^{\nu}}{\partial x'^{\mu}} = \delta^{\nu}_{\mu} - \partial_{\mu}\left(\omega_a \frac{\delta x^{\nu}}{\delta \omega_a}\right)$$
$$\left|\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right| = 1 + \partial_{\mu}\left(\omega_a \frac{\delta x^{\mu}}{\delta \omega_a}\right)$$

- The transformed action is then given by

$$S'\left[\Phi'\right] = \int d^d x \left(1 + \partial_{\mu}\left(\omega_a \frac{\delta x^{\mu}}{\delta \omega_a}\right)\right) \times$$
$$\mathcal{L}\left(\Phi + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}, \left[\delta^{\nu}_{\mu} - \partial_{\mu}\left(\omega_a \frac{\delta x^{\nu}}{\delta \omega_a}\right)\right]\left(\partial_{\nu}\Phi + \partial_{\nu}\left[\omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}\right]\right)\right)$$

# QFT: Noether Theorem

- The Jacobian and Jacobian matrix of the coordinate transformation is given by

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \delta^{\mu}_{\nu} + \partial_{\nu}\left(\omega_a \frac{\delta x^{\mu}}{\delta \omega_a}\right)$$
$$\frac{\partial x^{\nu}}{\partial x'^{\mu}} = \delta^{\nu}_{\mu} - \partial_{\mu}\left(\omega_a \frac{\delta x^{\nu}}{\delta \omega_a}\right)$$
$$\left|\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right| = 1 + \partial_{\mu}\left(\omega_a \frac{\delta x^{\mu}}{\delta \omega_a}\right)$$

- The transformed action is then given by

$$S'\left[\Phi'\right] = \int d^d x \left(1 + \partial_{\mu}\left(\omega_a \frac{\delta x^{\mu}}{\delta \omega_a}\right)\right) \times$$
$$\mathcal{L}\left(\Phi + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}, \left[\delta^{\nu}_{\mu} - \partial_{\mu}\left(\omega_a \frac{\delta x^{\nu}}{\delta \omega_a}\right)\right]\left(\partial_{\nu}\Phi + \partial_{\nu}\left[\omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}\right]\right)\right)$$

# QFT: Noether Theorem

- The divergence of the Noether current is then defined in terms of the difference between the transformed and original actions as

$$S' [\Phi'] - S [\Phi] \stackrel{\text{def.}}{=} \int d^d x \omega^a \partial_\mu j^\mu_a$$

- By replacing the form of the actions, and expanding to linear order in the transformations, we find that

$$j^\mu_a = \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} \partial_\nu \Phi - \delta^\mu_\nu \mathcal{L} \right) \frac{\delta x^\nu}{\delta \omega^a} - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} \frac{\delta \mathcal{F}}{\delta \omega^a}$$

# QFT: Noether Theorem

- The divergence of the Noether current is then defined in terms of the difference between the transformed and original actions as

$$S'\left[\Phi'\right] - S\left[\Phi\right] \stackrel{\text{def.}}{=} \int d^d x \omega^a \partial_\mu j^\mu_a$$

- By replacing the form of the actions, and expanding to linear order in the transformations, we find that

$$j^\mu_a = \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)} \partial_\nu \Phi - \delta^\mu_\nu \mathcal{L}\right) \frac{\delta x^\nu}{\delta \omega^a} - \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)} \frac{\delta \mathcal{F}}{\delta \omega^a}$$

# QFT: Energy-momentum tensor

- We consider translations as an example of transformation. Then, we have that

$$x'^{\lambda} = x^{\lambda} + \epsilon^{\lambda}$$
$$\Phi'(x') = \Phi(x)$$
$$\frac{\delta x^{\nu}}{\delta \epsilon^{\lambda}} = \delta^{\nu}_{\lambda} \; ; \; \frac{\delta \mathcal{F}}{\delta \epsilon^{\lambda}} = 0$$

- We define the E-M tensor as the corresponding Noether current, given by

$$T^{\mu\lambda} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\Phi)} \partial^{\lambda}\Phi - g^{\mu\lambda}\mathcal{L}$$

# QFT: Energy-momentum tensor

- We consider translations as an example of transformation. Then, we have that

$$x'^{\lambda} = x^{\lambda} + \epsilon^{\lambda}$$
$$\Phi'\left(x'\right) = \Phi\left(x\right)$$
$$\frac{\delta x^{\nu}}{\delta \epsilon^{\lambda}} = \delta^{\nu}_{\lambda} \; ; \; \frac{\delta \mathcal{F}}{\delta \epsilon^{\lambda}} = 0$$

- We define the E-M tensor as the corresponding Noether current, given by

$$T^{\mu\lambda} = \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\Phi\right)}\partial^{\lambda}\Phi - g^{\mu\lambda}\mathcal{L}$$

# Example: E-M tensor of real scalar field

- We compute the E-M tensor of a real scalar field. The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu \Phi \partial^\mu \Phi + m^2 \Phi^2\right)$$

- And the corresponding E-M tensor is

$$T^{\mu\nu} = \frac{1}{2}\left(\partial_\lambda \Phi \partial^\lambda \Phi + m^2 \Phi^2\right)\eta^{\mu\nu} - 2\partial^\mu \Phi \partial^\nu \Phi$$

- The 4-divergence of the E-M tensor is zero on-shell (when the Klein-Gordon equation of motion is satisfied):

$$\left(\partial_\mu \partial^\mu - m^2\right)\Phi = 0 \rightarrow \partial_\mu T^{\mu\nu} = 0$$

- Then, the E-M tensor is conserved and the theory has translation symmetry.

# Example: E-M tensor of real scalar field

- We compute the E-M tensor of a real scalar field. The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu \Phi \partial^\mu \Phi + m^2 \Phi^2\right)$$

- And the corresponding E-M tensor is

$$T^{\mu\nu} = \frac{1}{2}\left(\partial_\lambda \Phi \partial^\lambda \Phi + m^2 \Phi^2\right)\eta^{\mu\nu} - 2\partial^\mu \Phi \partial^\nu \Phi$$

- The 4-divergence of the E-M tensor is zero on-shell (when the Klein-Gordon equation of motion is satisfied):

$$\left(\partial_\mu \partial^\mu - m^2\right)\Phi = 0 \rightarrow \partial_\mu T^{\mu\nu} = 0$$

- Then, the E-M tensor is conserved and the theory has translation symmetry.

# Example: E-M tensor of real scalar field

- We compute the E-M tensor of a real scalar field. The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu \Phi \partial^\mu \Phi + m^2 \Phi^2\right)$$

- And the corresponding E-M tensor is

$$T^{\mu\nu} = \frac{1}{2}\left(\partial_\lambda \Phi \partial^\lambda \Phi + m^2 \Phi^2\right)\eta^{\mu\nu} - 2\partial^\mu \Phi \partial^\nu \Phi$$

- The 4-divergence of the E-M tensor is zero on-shell (when the Klein-Gordon equation of motion is satisfied):

$$\left(\partial_\mu \partial^\mu - m^2\right)\Phi = 0 \rightarrow \partial_\mu T^{\mu\nu} = 0$$

- Then, the E-M tensor is conserved and the theory has translation symmetry.

## Example: E-M tensor of real scalar field

- We compute the E-M tensor of a real scalar field. The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu \Phi \partial^\mu \Phi + m^2 \Phi^2\right)$$

- And the corresponding E-M tensor is

$$T^{\mu\nu} = \frac{1}{2}\left(\partial_\lambda \Phi \partial^\lambda \Phi + m^2 \Phi^2\right)\eta^{\mu\nu} - 2\partial^\mu \Phi \partial^\nu \Phi$$

- The 4-divergence of the E-M tensor is zero on-shell (when the Klein-Gordon equation of motion is satisfied):

$$\left(\partial_\mu \partial^\mu - m^2\right)\Phi = 0 \rightarrow \partial_\mu T^{\mu\nu} = 0$$

- Then, the E-M tensor is conserved and the theory has translation symmetry.

# QFT: Ward Identities

- Correlation function written in the (Euclidean) path integral formulation. Transformed field as renamed integration variable:

$$X \stackrel{\text{def.}}{=} \Phi(x_1)\cdots\Phi(x_n)$$
$$\langle X\rangle = \frac{1}{Z}\int [D\Phi]\,\Phi(x_1)\cdots\Phi(x_n)\,e^{-S[\Phi]}$$
$$\langle X\rangle = \frac{1}{Z}\int \left[D\Phi'\right]\,\Phi'(x_1)\cdots\Phi'(x_n)\,e^{-S[\Phi']}$$

- Definition of Noether current and transformed field product

$$S\left[\Phi'\right] = S[\Phi] + \int d^d x \omega^a \partial_\mu j^\mu_a$$
$$X + \delta X = \Phi'(x_1)\cdots\Phi'(x_n)$$

# QFT: Ward Identities

- Correlation function written in the (Euclidean) path integral formulation. Transformed field as renamed integration variable:

$$X \overset{\text{def.}}{=} \Phi(x_1)\cdots\Phi(x_n)$$
$$\langle X\rangle = \frac{1}{Z}\int [D\Phi]\,\Phi(x_1)\cdots\Phi(x_n)\,e^{-S[\Phi]}$$
$$\langle X\rangle = \frac{1}{Z}\int \left[D\Phi'\right]\Phi'(x_1)\cdots\Phi'(x_n)\,e^{-S[\Phi']}$$

- Definition of Noether current and transformed field product

$$S\left[\Phi'\right] = S[\Phi] + \int d^d x \omega^a \partial_\mu j_a^\mu$$
$$X + \delta X = \Phi'(x_1)\cdots\Phi'(x_n)$$

# QFT: Ward Identities

- Replacing into correlator definition to relate the transformtion of the correlator to the Noether current:

$$\langle X \rangle = \frac{1}{Z} \int [D\Phi] \left( X + \delta X \right) e^{-\left( S[\Phi] + \int d^d x \omega^a \partial_\mu j^\mu_a \right)}$$

$$\langle X \rangle \simeq \frac{1}{Z} \int [D\Phi] \left( X + \delta X \right) \left( 1 - \int d^d x \omega^a \partial_\mu j^\mu_a \right) e^{-S[\Phi]}$$

$$\langle X \rangle \simeq \langle X \rangle + \langle \delta X \rangle - \left\langle X \int d^d x \omega^a \partial_\mu j^\mu_a \right\rangle$$

$$\langle \delta X \rangle = \int d^d x \partial_\mu \left\langle j^\mu_a X \right\rangle \omega^a$$

# QFT: Ward Identities

- Transformation of correlator written from the definition of the generator using the product rule

$$\Phi'(x) = \Phi(x) - i\omega^a G_a \Phi(x)$$

$$\delta\Phi(x) = -i\omega^a G_a \Phi(x)$$

$$\delta X = -\sum_{i=1}^{n} \left(\Phi(x_1) \cdots [iG_a \Phi(x_i)] \cdots \Phi(x_n)\right) \omega^a$$

$$\langle \delta X \rangle = -\int d^d x \omega^a \sum_{i=1}^{n} \langle \Phi(x_1) \cdots [iG_a \Phi(x_i)] \cdots \Phi(x_n) \rangle \delta(x - x_i)$$

- Ward identity obtained by comparing both expressions:

$$\partial_\mu \langle j_a^\mu X \rangle = -\sum_{i=1}^{n} \langle \Phi(x_1) \cdots [iG_a \Phi(x_i)] \cdots \Phi(x_n) \rangle \delta(x - x_i)$$

# QFT: Ward Identities

- Transformation of correlator written from the definition of the generator using the product rule

$$\Phi'(x) = \Phi(x) - i\omega^a G_a \Phi(x)$$
$$\delta\Phi(x) = -i\omega^a G_a \Phi(x)$$
$$\delta X = -\sum_{i=1}^{n} \left(\Phi(x_1) \cdots \left[iG_a \Phi(x_i)\right] \cdots \Phi(x_n)\right) \omega^a$$
$$\langle \delta X \rangle = -\int d^d x \omega^a \sum_{i=1}^{n} \langle \Phi(x_1) \cdots \left[iG_a \Phi(x_i)\right] \cdots \Phi(x_n) \rangle \, \delta(x - x_i)$$

- Ward identity obtained by comparing both expressions:

$$\partial_\mu \langle j_a^\mu X \rangle = -\sum_{i=1}^{n} \langle \Phi(x_1) \cdots \left[iG_a \Phi(x_i)\right] \cdots \Phi(x_n) \rangle \, \delta(x - x_i)$$

# Conformal transformations

Di Francesco, Mathieu and Senechal, *Conformal Field Theory*

- Conformal transformation defined such that metric is rescaled. I.e.,

$$g'_{\mu\nu}\left(x'\right) = \Lambda\left(x\right) g_{\mu\nu}\left(x\right)$$

- For infinitesimal coordinate transformation, metric transforms as

$$\begin{aligned}
x'^{\mu} &= x^{\mu} + \epsilon^{\mu}\left(x\right) \\
g_{\mu'\nu'} &= \frac{\partial x^{\lambda}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}} g_{\lambda\sigma} \\
\frac{\partial x^{\mu}}{\partial x'^{\nu}} &= \delta^{\mu}_{\nu} - \partial_{\nu}\epsilon^{\mu}\left(x\right) \\
g'_{\mu\nu} &\rightarrow g_{\mu\nu} - \left(\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu}\right)
\end{aligned}$$

# Conformal transformations

Di Francesco, Mathieu and Senechal, *Conformal Field Theory*

- Conformal transformation defined such that metric is rescaled. I.e.,

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x)$$

- For infinitesimal coordinate transformation, metric transforms as

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x)$$

$$g_{\mu'\nu'} = \frac{\partial x^{\lambda}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} g_{\lambda\sigma}$$

$$\frac{\partial x^{\mu}}{\partial x'^{\nu}} = \delta^{\mu}_{\nu} - \partial_{\nu}\epsilon^{\mu}(x)$$

$$g'_{\mu\nu} \rightarrow g_{\mu\nu} - (\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu})$$

# Conformal transformations

- Proportionality between original and transformed metric fixes transformation to satisfy

$$(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \frac{2}{d} \partial_\lambda \epsilon^\lambda g_{\mu\nu}$$

- General solution is given by

$$\epsilon^\mu = a^\mu + \alpha x^\mu + m^\mu{}_\nu x^\nu + \left(2\left(x \cdot b\right) x^\mu - b^\mu \left(x^2\right)\right)$$

- Translations, Dilatations, Rotations and Special Conformal Transformations.

# Conformal transformations

- Proportionality between original and transformed metric fixes transformation to satisfy

$$(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \frac{2}{d} \partial_\lambda \epsilon^\lambda g_{\mu\nu}$$

- General solution is given by

$$\epsilon^\mu = a^\mu + \alpha x^\mu + m^\mu{}_\nu x^\nu + \left(2\left(x \cdot b\right) x^\mu - b^\mu \left(x^2\right)\right)$$

- Translations, Dilatations, Rotations and Special Conformal Transformations.

# Conformal transformations

- Proportionality between original and transformed metric fixes transformation to satisfy

$$(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \frac{2}{d} \partial_\lambda \epsilon^\lambda g_{\mu\nu}$$

- General solution is given by

$$\epsilon^\mu = a^\mu + \alpha x^\mu + m^\mu{}_\nu x^\nu + \left(2\left(x \cdot b\right) x^\mu - b^\mu \left(x^2\right)\right)$$

- Translations, Dilatations, Rotations and Special Conformal Transformations.

# Generators and conformal algebra

- Define generators such that they implement the transformation as

$$x'^{\mu} = (1 + i\omega^a G_a)\, x^{\mu}$$

- One obtains the following generators:

$$\begin{aligned} P_\mu &= -i\partial_\mu \\ D &= -ix^\mu \partial_\mu \\ L_{\mu\nu} &= i\,(x_\mu \partial_\nu - x_\nu \partial_\mu) \\ K_\mu &= -i\left(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu\right) \end{aligned}$$

# Generators and conformal algebra

- Define generators such that they implement the transformation as

$$x'^{\mu} = (1 + i\omega^a G_a)\, x^{\mu}$$

- One obtains the following generators:

$$\begin{aligned} P_\mu &= -i\partial_\mu \\ D &= -ix^\mu\partial_\mu \\ L_{\mu\nu} &= i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) \\ K_\mu &= -i\left(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu\right) \end{aligned}$$

# Generators and conformal algebra

- Conformal algebra given by

$$
\begin{aligned}
[D, P_\mu] &= iP_\mu \\
[D, K_\mu] &= -iK_\mu \\
[K_\mu, P_\nu] &= 2i\left(\eta_{\mu\nu} D - L_{\mu\nu}\right) \\
[K_\rho, L_{\mu\nu}] &= i\left(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu\right) \\
[P_\rho, L_{\mu\nu}] &= i\left(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu\right) \\
[L_{\mu\nu}, L_{\rho\sigma}] &= i\begin{pmatrix} \eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} \\ -\eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho} \end{pmatrix}
\end{aligned}
$$

# Quasi-primary fields

- For fields that have a particular spin and conformal dimension (irreps), transformation given by

$$\begin{aligned} P_\mu \Phi(x) &= -i\partial_\mu \Phi(x) \\ D\Phi(x) &= -i\left(x^\mu \partial_\mu + \Delta\right)\Phi(x) \\ L_{\mu\nu}\Phi(x) &= \left[i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) + S_{\mu\nu}\right]\Phi(x) \\ K_\mu \Phi(x) &= \left[-i\left(2x_\mu\left(x^\nu \partial_\nu + \Delta\right) - x^2 \partial_\mu\right) - x^\nu S_{\mu\nu}\right]\Phi(x) \end{aligned}$$

- For spin zero, transformation written as

$$S_{\mu\nu} = 0 \to \Phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta}{d}} \Phi(x)$$

- Definition of quasi-primary fields.

## Quasi-primary fields

- For fields that have a particular spin and conformal dimension (irreps), transformation given by

$$
\begin{aligned}
P_\mu \Phi(x) &= -i\partial_\mu \Phi(x) \\
D\Phi(x) &= -i\left(x^\mu \partial_\mu + \Delta\right)\Phi(x) \\
L_{\mu\nu}\Phi(x) &= \left[i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) + S_{\mu\nu}\right]\Phi(x) \\
K_\mu \Phi(x) &= \left[-i\left(2x_\mu\left(x^\nu \partial_\nu + \Delta\right) - x^2 \partial_\mu\right) - x^\nu S_{\mu\nu}\right]\Phi(x)
\end{aligned}
$$

- For spin zero, transformation written as

$$
S_{\mu\nu} = 0 \to \Phi'\left(x'\right) = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta}{d}} \Phi(x)
$$

- Definition of quasi-primary fields.

# Quasi-primary fields

- For fields that have a particular spin and conformal dimension (irreps), transformation given by

$$\begin{aligned} P_\mu \Phi(x) &= -i\partial_\mu \Phi(x) \\ D\Phi(x) &= -i\left(x^\mu \partial_\mu + \Delta\right)\Phi(x) \\ L_{\mu\nu}\Phi(x) &= \left[i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) + S_{\mu\nu}\right]\Phi(x) \\ K_\mu \Phi(x) &= \left[-i\left(2x_\mu\left(x^\nu \partial_\nu + \Delta\right) - x^2 \partial_\mu\right) - x^\nu S_{\mu\nu}\right]\Phi(x) \end{aligned}$$

- For spin zero, transformation written as

$$S_{\mu\nu} = 0 \to \Phi'\left(x'\right) = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta}{d}} \Phi(x)$$

- Definition of quasi-primary fields.

# Noether currents and Ward identities

- Noether currents for translation, dilatation and rotation written in terms of E-M tensor as

$$j_P^{\mu} = T^{\mu}{}_{\nu}$$
$$j_D^{\mu} = T^{\mu}{}_{\nu} x^{\nu}$$
$$j_M^{\mu} = T^{\mu\nu} x^{\rho} - T^{\mu\rho} x^{\nu}$$

- Corresponding Ward identities given by

$$\partial_{\mu} \langle T^{\mu}{}_{\nu} X \rangle = -\sum_i \delta(x - x_i) \frac{\partial}{\partial x_i^{\nu}} \langle X \rangle$$
$$\langle (T^{\rho\nu} - T^{\nu\rho}) X \rangle = -i \sum_i \delta(x - x_i) S_i^{\nu\rho} \langle X \rangle$$
$$\langle T^{\mu}{}_{\mu} X \rangle = -\sum_i \delta(x - x_i) \Delta_i \langle X \rangle$$

# Noether currents and Ward identities

- Noether currents for translation, dilatation and rotation written in terms of E-M tensor as

$$j_P^\mu = T^\mu{}_\nu$$
$$j_D^\mu = T^\mu{}_\nu x^\nu$$
$$j_M^\mu = T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu$$

- Corresponding Ward identities given by

$$\partial_\mu \langle T^\mu{}_\nu X \rangle = -\sum_i \delta(x - x_i) \frac{\partial}{\partial x_i^\nu} \langle X \rangle$$
$$\langle (T^{\rho\nu} - T^{\nu\rho}) X \rangle = -i \sum_i \delta(x - x_i) S_i^{\nu\rho} \langle X \rangle$$
$$\langle T^\mu{}_\mu X \rangle = -\sum_i \delta(x - x_i) \Delta_i \langle X \rangle$$

# Correlation functions

- By definition, correlator of 2 quasi-primary fields satisfies

$$\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta_1}{d}} \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta_2}{d}} \langle \phi_1(x_1')\,\phi_2(x_2')\rangle$$

- Invariance under rotations and translations fixes

$$\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = f(|x_1 - x_2|)$$

- For $x' = \lambda x$:

$$\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2} \langle \phi_1(\lambda x_1)\,\phi_2(\lambda x_2)\rangle$$

- Invariance under dilatation fixes

$$\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}$$

## Correlation functions

- By definition, correlator of 2 quasi-primary fields satisfies

$$\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta_1}{d}} \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta_2}{d}} \langle \phi_1(x_1')\,\phi_2(x_2')\rangle$$

- Invariance under rotations and translations fixes

$$\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = f(|x_1 - x_2|)$$

- For $x' = \lambda x$:

$$\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2} \langle \phi_1(\lambda x_1)\,\phi_2(\lambda x_2)\rangle$$

- Invariance under dilatation fixes

$$\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}$$

# Correlation functions

- By definition, correlator of 2 quasi-primary fields satisfies

$$\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta_1}{d}} \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta_2}{d}} \langle \phi_1(x_1')\,\phi_2(x_2')\rangle$$

- Invariance under rotations and translations fixes

$$\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = f(|x_1 - x_2|)$$

- For $x' = \lambda x$:

$$\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2} \langle \phi_1(\lambda x_1)\,\phi_2(\lambda x_2)\rangle$$

- Invariance under dilatation fixes

$$\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}$$

# Correlation functions

- By definition, correlator of 2 quasi-primary fields satisfies

$$\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta_1}{d}} \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta_2}{d}} \langle \phi_1(x_1')\,\phi_2(x_2')\rangle$$

- Invariance under rotations and translations fixes

$$\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = f(|x_1 - x_2|)$$

- For $x' = \lambda x$:

$$\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2} \langle \phi_1(\lambda x_1)\,\phi_2(\lambda x_2)\rangle$$

- Invariance under dilatation fixes

$$\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}$$

# Correlation functions

- Finally, invariance under special conformal transformations fixes conformal dimensions to be equal:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \begin{cases} \frac{C_{12}}{|x_1-x_2|^{2\Delta_1}} \text{ if } \Delta_1 = \Delta_2 \\ 0 \text{ if } \Delta_1 \neq \Delta_2 \end{cases}$$

- Correlator of three quasi-primaries also fixed by conformal symmetry:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{13}^{\Delta_3+\Delta_1-\Delta_2}}$$

# Correlation functions

- Finally, invariance under special conformal transformations fixes conformal dimensions to be equal:

$$\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = \begin{cases} \frac{C_{12}}{|x_1-x_2|^{2\Delta_1}} \text{ if } \Delta_1 = \Delta_2 \\ 0 \text{ if } \Delta_1 \neq \Delta_2 \end{cases}$$

- Correlator of three quasi-primaries also fixed by conformal symmetry:

$$\langle \phi_1(x_1)\,\phi_2(x_2)\,\phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}\,x_{23}^{\Delta_2+\Delta_3-\Delta_1}\,x_{13}^{\Delta_3+\Delta_1-\Delta_2}}$$

## Conformal transformations

Di Francesco, Mathieu and Senechal, *Conformal Field Theory*

Belavin, Polyakov and Zamolodchikov, Nucl. Phys. B 241, 333.

- Consider Euclidean (flat) metric and holomorphic coordinates defined by

$$ds^2 = dx^2 + dy^2$$
$$z = x + iy \; ; \; \bar{z} = x - iy$$
$$ds^2 = dzd\bar{z}$$

- Conformal transformation of coordinates requires rescaling of metric as

$$dzd\bar{z} = \rho\left(z', \bar{z}'\right) dz' d\bar{z}'$$

- Solution given by generic holomorphic (and anti-holomorphic) transformation:

$$z' = \zeta(z) \; ; \; \bar{z}' = \bar{\zeta}(\bar{z})$$

# Conformal transformations

Di Francesco, Mathieu and Senechal, *Conformal Field Theory*

Belavin, Polyakov and Zamolodchikov, Nucl. Phys. B 241, 333.

- Consider Euclidean (flat) metric and holomorphic coordinates defined by

$$ds^2 = dx^2 + dy^2$$
$$z = x + iy \; ; \; \bar{z} = x - iy$$
$$ds^2 = dz d\bar{z}$$

- Conformal transformation of coordinates requires rescaling of metric as

$$dz d\bar{z} = \rho \left(z', \bar{z}'\right) dz' d\bar{z}'$$

- Solution given by generic holomorphic (and anti-holomorphic) transformation:

$$z' = \zeta(z) \; ; \; \bar{z}' = \bar{\zeta}(\bar{z})$$

# Conformal transformations

Di Francesco, Mathieu and Senechal, *Conformal Field Theory*

Belavin, Polyakov and Zamolodchikov, Nucl. Phys. B 241, 333.

- Consider Euclidean (flat) metric and holomorphic coordinates defined by

$$ds^2 = dx^2 + dy^2$$
$$z = x + iy \ ; \ \bar{z} = x - iy$$
$$ds^2 = dz d\bar{z}$$

- Conformal transformation of coordinates requires rescaling of metric as

$$dz d\bar{z} = \rho\left(z', \bar{z}'\right) dz' d\bar{z}'$$

- Solution given by generic holomorphic (and anti-holomorphic) transformation:

$$z' = \zeta\left(z\right) \ ; \ \bar{z}' = \bar{\zeta}\left(\bar{z}\right)$$

# Generators and Witt algebra

- Holomorphic function admits Laurent expansion. Generators of transformation obey Witt algebra, given by

$$z' = z + \epsilon(z)$$

$$\epsilon(z) = \sum_{n=-\infty}^{\infty} \epsilon_n z^{n+1}$$

$$z' = (1 - \epsilon_n L_n) z$$

$$L_n = -z^{n+1} \partial_z$$

$$[L_n, L_m] = (n - m) L_{n+m}$$

# Quasi-primary fields

- Quasi-primary fields of specific conformal dimension (and spin) transform as

$$\phi'\left(z',\bar{z}'\right) = \left|\frac{dz'}{dz}\right|^{2h} \left|\frac{d\bar{z}'}{d\bar{z}}\right|^{2\bar{h}} \phi\left(z,\bar{z}\right)$$

$$h = \frac{1}{2}(\Delta + s) \; ; \; \bar{h} = \frac{1}{2}(\Delta - s)$$

- In 2D, spin of quasi-primaries can be non-zero and it modifies the holomorphic and anti-holomorphic conformal dimensions.

# Quasi-primary fields

- Quasi-primary fields of specific conformal dimension (and spin) transform as

$$\phi'\left(z',\bar{z}'\right) = \left|\frac{dz'}{dz}\right|^{2h} \left|\frac{d\bar{z}'}{d\bar{z}}\right|^{2\bar{h}} \phi\left(z,\bar{z}\right)$$

$$h = \frac{1}{2}\left(\Delta + s\right)\ ;\ \bar{h} = \frac{1}{2}\left(\Delta - s\right)$$

- In 2D, spin of quasi-primaries can be non-zero and it modifies the holomorphic and anti-holomorphic conformal dimensions.

# Ward identities

- Dirac delta has representation in 2D given by

$$\delta\left(x - w_i\right) = \frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z - w_i} = \frac{1}{\pi}\partial_{z}\frac{1}{\bar{z} - w_i}$$

- Conformal Ward identity becomes

$$\partial_{\bar{z}}\left(\langle T(z) X\rangle - \sum_{i=1}^{n}\left[\frac{1}{z_i - w_i}\partial_{w_i}\langle X\rangle + \frac{h_i}{\left(z_i - w_i\right)^2}\langle X\rangle\right]\right) = 0$$

$$\partial_{z}\left(\langle \overline{T}(\bar{z}) X\rangle - \sum_{i=1}^{n}\left[\frac{1}{\bar{z}_i - \overline{w}_i}\partial_{\overline{w}_i}\langle X\rangle + \frac{\bar{h}_i}{\left(\bar{z}_i - \overline{w}_i\right)^2}\langle X\rangle\right]\right) = 0$$

- Definition of holomorphic and anti-holomorphic parts of E-M tensor:

$$T(z) = -2\pi T_{zz}$$

$$\overline{T}(\bar{z}) = -2\pi T_{\bar{z}\bar{z}}$$

# Ward identities

- Dirac delta has representation in 2D given by

$$\delta\left(x-w_i\right)=\frac{1}{\pi}\partial_{\overline{z}}\frac{1}{z-w_i}=\frac{1}{\pi}\partial_{z}\frac{1}{\overline{z}-w_i}$$

- Conformal Ward identity becomes

$$\partial_{\overline{z}}\left(\left\langle T\left(z\right)X\right\rangle-\sum_{i=1}^{n}\left[\frac{1}{z_i-w_i}\partial_{w_i}\left\langle X\right\rangle+\frac{h_i}{\left(z_i-w_i\right)^2}\left\langle X\right\rangle\right]\right)=0$$

$$\partial_{z}\left(\left\langle \overline{T}\left(\overline{z}\right)X\right\rangle-\sum_{i=1}^{n}\left[\frac{1}{\overline{z}_i-\overline{w}_i}\partial_{\overline{w}_i}\left\langle X\right\rangle+\frac{\overline{h}_i}{\left(\overline{z}_i-\overline{w}_i\right)^2}\left\langle X\right\rangle\right]\right)=0$$

- Definition of holomorphic and anti-holomorphic parts of E-M tensor:

$$T\left(z\right)=-2\pi T_{zz}$$

$$\overline{T}\left(\overline{z}\right)=-2\pi T_{\overline{z}\overline{z}}$$

# Ward identities

- Dirac delta has representation in 2D given by

$$\delta\left(x-w_i\right)=\frac{1}{\pi}\partial_{\overline{z}}\frac{1}{z-w_i}=\frac{1}{\pi}\partial_{z}\frac{1}{\overline{z}-w_i}$$

- Conformal Ward identity becomes

$$\partial_{\overline{z}}\left(\left\langle T\left(z\right)X\right\rangle-\sum_{i=1}^{n}\left[\frac{1}{z_i-w_i}\partial_{w_i}\left\langle X\right\rangle+\frac{h_i}{\left(z_i-w_i\right)^2}\left\langle X\right\rangle\right]\right)=0$$

$$\partial_{z}\left(\left\langle \overline{T}\left(\overline{z}\right)X\right\rangle-\sum_{i=1}^{n}\left[\frac{1}{\overline{z}_i-\overline{w}_i}\partial_{\overline{w}_i}\left\langle X\right\rangle+\frac{\overline{h}_i}{\left(\overline{z}_i-\overline{w}_i\right)^2}\left\langle X\right\rangle\right]\right)=0$$

- Definition of holomorphic and anti-holomorphic parts of E-M tensor:

$$T\left(z\right)=-2\pi T_{zz}$$

$$\overline{T}\left(\overline{z}\right)=-2\pi T_{\overline{z}\overline{z}}$$

# Holomorphic energy-momentum tensor and Virasoro algebra

- E-M tensor written in terms of generators of conformal algebra as

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{\widehat{L}_n}{z^{n+2}} \; ; \; \overline{T}(\overline{z}) = \sum_{n=-\infty}^{\infty} \frac{\overline{\widehat{L}}_n}{\overline{z}^{n+2}}$$

- In quantum case, there is central extension. Virasoro algebra:

$$\left[\widehat{L}_m, \widehat{L}_n\right] = (n-m)\,\widehat{L}_{n+m} + \frac{1}{12} c \left(n^3 - n\right) \delta_{n+m,0}$$

- Transformation on quasi-primaries implemented by Virasoro generators:

$$\left[\widehat{L}_m, \phi(z)\right] = z^{m+1} \frac{\partial}{\partial z} \phi(z) + h(m+1)\, z^m \phi(z)$$

# Holomorphic energy-momentum tensor and Virasoro algebra

- E-M tensor written in terms of generators of conformal algebra as

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{\widehat{L}_n}{z^{n+2}} \; ; \; \overline{T}(\overline{z}) = \sum_{n=-\infty}^{\infty} \frac{\overline{\widehat{L}}_n}{\overline{z}^{n+2}}$$

- In quantum case, there is central extension. Virasoro algebra:

$$\left[\widehat{L}_m, \widehat{L}_n\right] = (n-m)\,\widehat{L}_{n+m} + \frac{1}{12} c\left(n^3 - n\right)\delta_{n+m,0}$$

- Transformation on quasi-primaries implemented by Virasoro generators:

$$\left[\widehat{L}_m, \phi(z)\right] = z^{m+1}\frac{\partial}{\partial z}\phi(z) + h(m+1)\,z^m\phi(z)$$

# Holomorphic energy-momentum tensor and Virasoro algebra

- E-M tensor written in terms of generators of conformal algebra as

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{\widehat{L}_n}{z^{n+2}} \; ; \; \overline{T}(\overline{z}) = \sum_{n=-\infty}^{\infty} \frac{\overline{\widehat{L}}_n}{\overline{z}^{n+2}}$$

- In quantum case, there is central extension. Virasoro algebra:

$$\left[\widehat{L}_m, \widehat{L}_n\right] = (n-m)\,\widehat{L}_{n+m} + \frac{1}{12} c\left(n^3 - n\right)\delta_{n+m,0}$$

- Transformation on quasi-primaries implemented by Virasoro generators:

$$\left[\widehat{L}_m, \phi(z)\right] = z^{m+1}\frac{\partial}{\partial z}\phi(z) + h\,(m+1)\,z^m\phi(z)$$

# Transformation of energy-momentum tensor and Casimir energy

- Under conformal transformation, E-M tensor transforms as

$$T(z) = \left(\frac{dz'}{dz}\right)^2 T(z') + \frac{c}{12}\{z', z\}$$

$$\{z', z\} = \frac{(d^3z'/dz^3)}{dz'/dz} - \frac{3}{2}\left(\frac{d^2z'/dz^2}{dz'/dz}\right)^2$$

- Schwarzian derivative gives rise to (vacuum) casimir energy.

# Transformation of energy-momentum tensor and Casimir energy

- Under conformal transformation, E-M tensor transforms as

$$T(z) = \left(\frac{dz'}{dz}\right)^2 T(z') + \frac{c}{12}\{z', z\}$$
$$\{z', z\} = \frac{(d^3z'/dz^3)}{dz'/dz} - \frac{3}{2}\left(\frac{d^2z'/dz^2}{dz'/dz}\right)^2$$

- Schwarzian derivative gives rise to (vacuum) casimir energy.

# Correlation functions

- Two-point function fixed by conformal symmetry:

$$\langle \phi_1(z_1,\bar{z}_1)\,\phi_2(z_2,\bar{z}_2)\rangle = \frac{C_{12}}{(z_1-z_2)^{2h}(\bar{z}_1-\bar{z}_2)^{2\bar{h}}}$$

- Three-point function fixed by conformal symmetry:

$$\langle \phi_1\phi_2\phi_3\rangle = \frac{C_{123}}{\left(\begin{array}{c}\left(z_{12}^{2(h_1+h_2-h_3)}\,z_{23}^{2(h_2+h_3-h_1)}\,z_{13}^{2(h_3+h_1-h_2)}\right)\times\\ \left(\bar{z}_{12}^{2(\bar{h}_1+\bar{h}_2-\bar{h}_3)}\,\bar{z}_{23}^{2(\bar{h}_2+\bar{h}_3-\bar{h}_1)}\,\bar{z}_{13}^{2(\bar{h}_3+\bar{h}_1-\bar{h}_2)}\right)\end{array}\right)}$$

# Correlation functions

- Two-point function fixed by conformal symmetry:

$$\langle \phi_1 (z_1, \bar{z}_1) \phi_2 (z_2, \bar{z}_2) \rangle = \frac{C_{12}}{(z_1 - z_2)^{2h} (\bar{z}_1 - \bar{z}_2)^{2\bar{h}}}$$

- Three-point function fixed by conformal symmetry:

$$\langle \phi_1 \phi_2 \phi_3 \rangle = \frac{C_{123}}{\left( \begin{array}{c} \left( z_{12}^{2(h_1+h_2-h_3)} z_{23}^{2(h_2+h_3-h_1)} z_{13}^{2(h_3+h_1-h_2)} \right) \times \\ \left( \bar{z}_{12}^{2\left(\bar{h}_1+\bar{h}_2-\bar{h}_3\right)} \bar{z}_{23}^{2\left(\bar{h}_2+\bar{h}_3-\bar{h}_1\right)} \bar{z}_{13}^{2\left(\bar{h}_3+\bar{h}_1-\bar{h}_2\right)} \right) \end{array} \right)}$$