

PAUTA CERTAMEN N° 3 PARTE 1
ECUACIONES DIFERENCIALES
INGENIERÍA CIVIL AGRÍCOLA

NOMBRE : _____

TIEMPO MÁXIMO : 1 HORA 20 MINUTOS

FECHA : Mi 03/12/25

Resuelva, usando series, la EDO:

a) $(1 - x) y'' - 2y' = 0$

(30 puntos)

Solución:

Sea $y = \sum_{n=0}^{\infty} c_n x^n$

Luego : $y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$; $y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$

Reemplazando lo anterior en la ecuación diferencial:

$$(1 - x) y'' - 2y' = 0 \Rightarrow y'' - x y'' - 2y' = 0$$
$$\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} - \sum_{n=2}^{\infty} c_n n(n-1) x^{n-1} - \sum_{n=1}^{\infty} 2 c_n n x^{n-1} = 0$$

Trabajemos la serie $\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$ de modo que la potencia de x sea $n-1$

Sea $n-2 = k-1$, luego $n = k+1$ y además si $n = 2$, entonces $k = 1$. Por tanto,

$$\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} = \sum_{k=1}^{\infty} c_{k+1} (k+1) k x^{k-1}$$

Si ahora volvemos a la variable n en la última expresión anterior, se tiene

$$\sum_{n=1}^{\infty} c_{n+1} (n+1) n x^{n-1} - \sum_{n=2}^{\infty} c_n n(n-1) x^{n-1} - \sum_{n=1}^{\infty} 2 c_n n x^{n-1} = 0 \Rightarrow$$
$$2c_2 + \sum_{n=2}^{\infty} c_{n+1} (n+1) n x^{n-1} - \sum_{n=2}^{\infty} c_n n(n-1) x^{n-1} - 2c_1 - \sum_{n=2}^{\infty} 2 c_n n x^{n-1} = 0 \Rightarrow$$
$$(2c_2 - 2c_1) + \sum_{n=2}^{\infty} c_{n+1} (n+1) n x^{n-1} - \sum_{n=2}^{\infty} c_n n(n-1) x^{n-1} - \sum_{n=2}^{\infty} 2 c_n n x^{n-1} = 0 \Rightarrow$$
$$(2c_2 - 2c_1) + \sum_{n=2}^{\infty} [c_{n+1} (n+1) n - c_n n(n-1) - 2 c_n n] x^{n-1} = 0 \Rightarrow$$
$$(2c_2 - 2c_1) = 0 \quad \text{y} \quad c_{n+1} (n+1) n - c_n n(n-1) - 2 c_n n = 0 \Rightarrow$$

$$(2c_2 - 2c_1) = 0 \quad \text{y} \quad c_{n+1}(n+1)n = c_n n(n-1) + 2c_n n = 0 \Rightarrow$$

$$c_2 = c_1 \quad \text{y} \quad c_{n+1}(n+1)n = n[(n-1) + 2]c_n \Rightarrow$$

$$c_2 = c_1 \quad \text{y} \quad c_{n+1}(n+1)n = n(n+1)c_n \Rightarrow c_2 = c_1 \quad \text{y} \quad c_{n+1} = c_n$$

$$c_{n+1} = c_n, \quad n = 2, 3, 4, \dots$$

$$n = 2 : c_3 = c_2 = c_1$$

$$n = 3 : c_4 = c_3 = c_1$$

$$n = 4 : c_5 = c_4 = c_1$$

$$n = 5 : c_6 = c_5 = c_1$$

Tenemos que

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \dots = \\ &c_0 + c_1 x + c_1 x^2 + c_1 x^3 + c_1 x^4 + c_1 x^5 + c_1 x^6 + \dots = \\ &c_0 + c_1 (x + x^2 + x^3 + x^4 + x^5 + x^6 + \dots) = c_0 + c_1 \sum_{n=1}^{\infty} x^n = c_0 + c_1 \frac{x}{1-x} \end{aligned}$$

Finalmente, $y(x) = c_0 + c_1 \frac{x}{1-x}$ $\square$

$$b) 4x y'' + 2y' + y = 0$$

(30 puntos)

Solución:

Dado que y'' está multiplicada por $4x$, usaremos la serie de Frobenius

$$\text{Sea } y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}$$

Reemplazando en la ecuación diferencial:

$$4x y'' + 2y' + y = 0 \Rightarrow$$

$$\sum_{n=0}^{\infty} 4c_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} 2c_n (n+r) x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

Para la serie infinita $\sum_{n=0}^{\infty} c_n x^{n+r}$ podemos hacer $k + r - 1 = n + r$, es decir,

$k - 1 = n$ y si $n = 0$, entonces $k = 1$. De donde $\sum_{n=0}^{\infty} c_n x^{n+r} = \sum_{k=1}^{\infty} c_{k-1} x^{k+r-1}$

Ahora si escribimos n en vez de k , se tiene que

$$\begin{aligned} \sum_{n=0}^{\infty} 4c_n(n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} 2c_n(n+r) x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} &= 0 \Rightarrow \\ \sum_{n=0}^{\infty} 4c_n(n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} 2c_n(n+r) x^{n+r-1} + \sum_{n=1}^{\infty} c_{n-1} x^{n+r-1} &= 0 \Rightarrow \\ 4c_0r(r-1) x^{r-1} + \sum_{n=1}^{\infty} 4c_n(n+r)(n+r-1) x^{n+r-1} + 2c_0r x^{r-1} + & \\ \sum_{n=1}^{\infty} 2c_n(n+r) x^{n+r-1} + \sum_{n=1}^{\infty} c_{n-1} x^{n+r-1} &= 0 \Rightarrow \end{aligned}$$

$$[4r(r-1) + 2r] c_0 x^{r-1} = 0 \quad \text{y}$$

$$\sum_{n=1}^{\infty} [4c_n(n+r)(n+r-1) + 2c_n(n+r) + c_{n-1}] x^{n+r-1} = 0 \Rightarrow c_0 \neq 0$$

$$4r(r-1) + 2r = 0 \quad \text{y} \quad 4c_n(n+r)(n+r-1) + 2c_n(n+r) + c_{n-1} = 0$$

Resolvamos la ecuación indicial $4r(r-1) + 2r = 0$

$$\begin{aligned} 4r(r-1) + 2r = 0 &\Rightarrow 4r^2 - 4r + 2r = 0 \Rightarrow 4r^2 - 2r = 0 \Rightarrow 2r(2r-1) = 0 \Rightarrow \\ r = 0 \quad \vee \quad r &= \frac{1}{2} \end{aligned}$$

Obtengamos la relación de recurrencia:

$$\begin{aligned} 4c_n(n+r)(n+r-1) + 2c_n(n+r) + c_{n-1} &= 0 \Rightarrow \\ c_n[4(n+r)(n+r-1) + 2(n+r)] + c_{n-1} &= 0 \Rightarrow c_n = -\frac{c_{n-1}}{2(n+r)(2n+2r-1)}, n = 1, 2, \dots \end{aligned}$$

$$\text{Para } r = \frac{1}{2} : c_n = -\frac{c_{n-1}}{2(n+\frac{1}{2})(2n+1-1)}, n = 1, 2, \dots$$

$$c_n = -\frac{c_{n-1}}{2n(2n+1)}, n = 1, 2, \dots$$

$$n = 1 : c_1 = -\frac{c_0}{2(3)} = -\frac{c_0}{3!}$$

$$n = 2 : c_2 = -\frac{c_1}{2(2)(5)} = \frac{c_0}{2(3)2(2)(5)} = \frac{c_0}{2(3)(4)(5)} = \frac{c_0}{5!}$$

$$n = 3 : c_3 = -\frac{c_2}{2(3)(7)} = -\frac{c_2}{(6)(7)} = -\frac{c_0}{2(3)(4)(5)(6)(7)} = -\frac{c_0}{7!}$$

$$\text{Observamos que } c_n = (-1)^n \frac{c_0}{(2n+1)!}$$

$$\text{Reemplazando en } y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}:$$

$$y(x) = \sum_{n=0}^{\infty} (-1)^n \frac{c_0}{(2n+1)!} x^{n+1/2} = c_0 \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{(2n+1)/2} =$$

$$c_0 \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} (\sqrt{x})^{(2n+1)} = c_0 \operatorname{sen}(\sqrt{x}), \quad x > 0$$

Para $r = 0$: $c_n = -\frac{c_{n-1}}{(2n-1)2n}, n = 1, 2, \dots$

$$n = 1 : c_1 = -\frac{c_0}{(1)2} = -\frac{c_0}{2!}$$

$$n = 2 : c_2 = -\frac{c_1}{(3)4} = \frac{c_0}{(1)2(3)4!} = \frac{c_0}{4!}$$

Ya observamos que, de modo similar a lo anterior, se tiene $c_n = (-1)^n \frac{c_0}{(2n)!}$

Reemplazando en $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$:

$$y(x) = \sum_{n=0}^{\infty} (-1)^n \frac{c_0}{(2n)!} x^n = c_0 \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} (\sqrt{x})^{2n} = c_0 \cos(\sqrt{x})$$

La solución final es una combinación lineal de las soluciones anteriores, es decir,

$$y(x) = k_1 \operatorname{sen}(\sqrt{x}) + k_2 \cos(\sqrt{x}), \text{ con } k_1 \text{ y } k_2 \text{ constantes reales y } x > 0. \quad \square$$

Observación: Una EDO se resuelve usando serie de potencias y la otra usando serie de Frobenius.