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PAUTA TAREA ECUACIONES DIFERENCIALES
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1) Use serie de potencias o de Frobenius, para resolver las siguientes ecuaciones diferenciales

a) $y'' + x\,y = 0$ (Ecuación de Airy)

Solución:

Sea $y(x) = \sum_{n=0}^{\infty} c_n\, x^n$

Luego

$y'(x) = \sum_{n=0}^{\infty} n\, c_n\, x^{n-1}$

$y''(x) = \sum_{n=0}^{\infty} n(n-1)\, c_n\, x^{n-2}$

Reemplazando en la ecuación diferencial

$$y'' + x\,y = 0 \Rightarrow \sum_{n=0}^{\infty} n(n-1)\, c_n\, x^{n-2} + x\sum_{n=0}^{\infty} c_n\, x^n = 0 \Rightarrow$$

$$\sum_{n=0}^{\infty} n(n-1)\, c_n\, x^{n-2} + \sum_{n=0}^{\infty} c_n\, x^{n+1} = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1)\, c_n\, x^{n-2} + \sum_{n=0}^{\infty} c_n\, x^{n+1} = 0 \Rightarrow$$

$$2c_2 + \sum_{n=3}^{\infty} n(n-1)\, c_n\, x^{n-2} + \sum_{n=0}^{\infty} c_n\, x^{n+1} = 0 \Rightarrow$$

$$2c_2 + \sum_{n=0}^{\infty} (n+3)(n-1+3)\, c_{n+3}\, x^{n-2+3} + \sum_{n=0}^{\infty} c_n\, x^{n+1} = 0 \Rightarrow$$

$$2c_2 + \sum_{n=0}^{\infty} (n+3)(n+2)\, c_{n+3}\, x^{n+1} + \sum_{n=0}^{\infty} c_n\, x^{n+1} = 0 \Rightarrow$$

$$2c_2 + \sum_{n=0}^{\infty} [(n+3)(n+2)\, c_{n+3} + c_n]\, x^{n+1} = 0 \Rightarrow$$

$2c_2 = 0$ y $(n+3)(n+2)\, c_{n+3} + c_n = 0 \Rightarrow$

$c_2 = 0$ y $c_{n+3} = -\frac{c_n}{(n+2)(n+3)},\ n = 0, 1, 2, ...$

$n = 0 \Rightarrow c_3 = -\frac{c_0}{2\cdot 3}$

$n = 1 \Rightarrow c_4 = -\frac{c_1}{3\cdot 4}$

$n = 2 \Rightarrow c_5 = -\frac{c_2}{4\cdot 5} = 0$

$n = 3 \Rightarrow c_6 = -\frac{c_3}{5\cdot 6} = \frac{c_0}{2\cdot 3\cdot 5\cdot 6}$

$n = 4 \Rightarrow c_7 = -\frac{c_4}{6 \cdot 7} = \frac{c_1}{3 \cdot 4 \cdot 6 \cdot 7}$
$n = 5 \Rightarrow c_8 = -\frac{c_5}{7 \cdot 8} = 0$
$n = 6 \Rightarrow c_9 = -\frac{c_6}{8 \cdot 9} = -\frac{c_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}$
$n = 7 \Rightarrow c_{10} = -\frac{c_7}{9 \cdot 10} = -\frac{c_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10}$
$n = 8 \Rightarrow c_{11} = -\frac{c_8}{10 \cdot 11} = 0$

Podemos continuar, pero ya podemos reempazar el $y(x) = \sum_{n=0}^{\infty} c_n x^n$

$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + c_8 x^8 +$
$c_9 x^9 + c_{10} x^{10} + c_{11} x^{11} + \ldots.$

$y(x) = c_0 + c_1 x - c_0 \frac{1}{2 \cdot 3} x^3 - c_1 \frac{1}{3 \cdot 4} x^4 + c_0 \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} x^6 + c_1 \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} x^7$
$- c_0 \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} x^9 - c_1 \frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} x^{10} + \ldots.$

$y(x) = c_0 y_1(x) + c_1 y_2(x)$, donde

$y_1(x) = 1 - \frac{1}{2 \cdot 3} x^3 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} x^6 - \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} x^9 + \ldots. =$
$1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2 \cdot 3 \cdots (3n-1)(3n)} x^{3n}$
$y_2(x) = x - \frac{1}{3 \cdot 4} x^4 + \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} x^7 - \frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} x^{10} + \ldots =$
$x + \sum_{n=1}^{\infty} \frac{(-1)^n}{3 \cdot 4 \cdots (3n)(3n+1)} x^{3n+1}$ ❑

b) $(x-3)\, y' + 2\, y = 0$

<u>Solución:</u>

Sea $y(x) = \sum_{n=0}^{\infty} c_n x^n$

Luego

$y'(x) = \sum_{n=0}^{\infty} n\, c_n x^{n-1}$

Reemplazando en la ecuación diferencial

$(x-3)\, y' + 2\, y = 0 \Rightarrow (x-3) \sum_{n=0}^{\infty} n\, c_n x^{n-1} + 2 \sum_{n=0}^{\infty} c_n x^n = 0 \Rightarrow$
$\sum_{n=0}^{\infty} n\, c_n x^n - \sum_{n=0}^{\infty} 3n\, c_n x^{n-1} + \sum_{n=0}^{\infty} 2c_n x^n = 0 \Rightarrow$
$\sum_{n=0}^{\infty} n\, c_n x^n - \sum_{n=1}^{\infty} 3n\, c_n x^{n-1} + \sum_{n=0}^{\infty} 2c_n x^n = 0 \Rightarrow$
$\sum_{n=0}^{\infty} n\, c_n x^n - \sum_{n=0}^{\infty} 3(n+1)\, c_{n+1} x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0 \Rightarrow$
$\sum_{n=0}^{\infty} [n\, c_n - 3(n+1)\, c_{n+1} + 2\, c_n]\, x^n = 0 \Rightarrow$
$n\, c_n - 3(n+1)\, c_{n+1} + 2\, c_n = 0 \Rightarrow c_n(n+2) - 3(n+1)\, c_{n+1} = 0 \Rightarrow$

$c_{n+1} = \frac{n+2}{3(n+1)}\, c_n,\ n = 0, 1, 2, ...$

$n = 0 \Rightarrow c_1 = \frac{2}{3} c_0$

$n = 1 \Rightarrow c_2 = \frac{3}{3\cdot 2}\, c_1 = \frac{3}{3\cdot 2}\frac{2}{3} c_0 = \frac{3}{3^2} c_0$

$n = 2 \Rightarrow c_3 = \frac{4}{3\cdot 3}\, c_2 = \frac{4}{3\cdot 3}\frac{3}{3^2} c_0 = \frac{4}{3^3} c_0$

$n = 3 \Rightarrow c_4 = \frac{5}{3\cdot 4}\, c_3 = \frac{5}{3\cdot 4}\frac{4}{3^3} c_0 = \frac{5}{3^4} c_0$

Ya se percibe el comportamiento de los coeficientes, por lo que

$c_n = \frac{n+1}{3^n} c_0,\ n = 0, 1, 2, ...$

La solución de la ecuación diferencial es

$y(x) = c_0 \sum_{n=0}^{\infty} \frac{n+1}{3^n}\, x^n =^{GeoGebra}\ c_0\ \frac{9}{x^2-6x+9} = c_0 \frac{9}{(x-3)^2}$ ❑

c) $x\, y'' - y' + 4x^3\, y = 0$

Solución:

Sea $y(x) = \sum_{n=0}^{\infty} c_n\, x^n$

Luego

$y'(x) = \sum_{n=0}^{\infty} n\, c_n\, x^{n-1}$

$y''(x) = \sum_{n=0}^{\infty} n(n-1)\, c_n\, x^{n-2}$

Reemplazando en la ecuación diferencial

$x\, y'' - y' + 4x^3\, y = 0 \Rightarrow \sum_{n=0}^{\infty} n(n-1)\, c_n\, x^{n-1} - \sum_{n=0}^{\infty} n\, c_n\, x^{n-1} + \sum_{n=0}^{\infty} 4c_n\, x^{n+3} = 0$

$\Rightarrow 2c_2\, x + 6c_3 x^2 + \sum_{n=4}^{\infty} n(n-1)\, c_n\, x^{n-1} - c_1 - 2c_2 x - 3c_3 x^2 - \sum_{n=4}^{\infty} n\, c_n\, x^{n-1}$

$+ \sum_{n=0}^{\infty} 4c_n\, x^{n+3} = 0$

$\Rightarrow 3c_3 x^2 + \sum_{n=4}^{\infty} n(n-1)\, c_n\, x^{n-1} - c_1 - \sum_{n=4}^{\infty} n\, c_n\, x^{n-1} + \sum_{n=0}^{\infty} 4c_n\, x^{n+3} = 0$

$\Rightarrow 3c_3 x^2 + \sum_{n=0}^{\infty} (n+4)(n+3)\, c_{n+4}\, x^{n+3} - c_1 - \sum_{n=0}^{\infty} (n+4)\, c_{n+4}\, x^{n+3}$

$+ \sum_{n=0}^{\infty} 4c_n\, x^{n+3} = 0$

$\Rightarrow 3c_3 x^2 + \sum_{n=0}^{\infty} \left([(n+4)(n+3) - (n+4)]\, c_{n+4} + 4c_n \right)\, x^{n+3} - c_1 = 0$

$\Rightarrow 3c_3 x^2 + \sum_{n=0}^{\infty} \left([(n+4)(n+2)]\, c_{n+4} + 4c_n \right)\, x^{n+3} - c_1 = 0$

$\Rightarrow c_1 = 0,\ c_3 = 0,\ [(n+4)(n+2)]\, c_{n+4} + 4c_n = 0$

$\Rightarrow c_{n+4} = -\frac{4c_n}{(n+2)(n+4)}, n = 0, 1, 2, ...$

$n = 0 \Rightarrow c_4 = -\frac{4c_0}{2\cdot 4}$

$n = 1 \Rightarrow c_5 = -\frac{4c_1}{3\cdot 5} = 0$

$n=2 \Rightarrow c_6 = -\frac{4c_2}{4\cdot 6}$
$n=3 \Rightarrow c_7 = -\frac{4c_3}{5\cdot 7} = 0$
$n=4 \Rightarrow c_8 = -\frac{4c_4}{6\cdot 8} = \frac{4^2 c_0}{2\cdot 4\cdot 6\cdot 8}$
$n=5 \Rightarrow c_9 = -\frac{4c_5}{7\cdot 9} = 0$
$n=6 \Rightarrow c_{10} = -\frac{4c_6}{8\cdot 10} = \frac{4^2 c_2}{4\cdot 6\cdot 8\cdot 10}$
$n=7 \Rightarrow c_{11} = -\frac{4c_7}{9\cdot 11} = 0$
$n=8 \Rightarrow c_{12} = -\frac{4c_8}{10\cdot 12} = -\frac{4^3 c_0}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12}$
$n=9 \Rightarrow c_{13} = -\frac{4c_9}{11\cdot 13} = 0$
$n=10 \Rightarrow c_{14} = -\frac{4c_{10}}{12\cdot 14} = -\frac{4^3 c_2}{4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14}$
Reemplazando en $y(x) = \sum_{n=0}^{\infty} c_n\, x^n$

$y(x) = c_0 + c_2 x^2 - \frac{4c_0}{2\cdot 4}x^4 - \frac{4c_2}{4\cdot 6}x^6 + \frac{4^2 c_0}{2\cdot 4\cdot 6\cdot 8}x^8 + \frac{4^2 c_2}{4\cdot 6\cdot 8\cdot 10}x^{10} - \frac{4^3 c_0}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12}x^{12}$
$- \frac{4^3 c_2}{4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14}x^{14} + ... =$
$c_0(1 - \frac{4}{2\cdot 4}x^4 + \frac{4^2}{2\cdot 4\cdot 6\cdot 8}x^8 - \frac{4^3}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12}x^{12} + ...) +$
$c_2(x^2 - \frac{4}{4\cdot 6}x^6 + \frac{4^2}{4\cdot 6\cdot 8\cdot 10}x^{10} - \frac{4^3}{4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14}x^{14} + ...)$

$y(x) = c_0\, y_1(x) + c_2 y_2(x)$
$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{4^n\,(-1)^n}{2\cdot 4\cdots(4n-2)(4n)} x^{4n} =^{GeoGebra} cos(x^2)$
$y_2(x) = 1 + \sum_{n=1}^{\infty} \frac{4^n\,(-1)^n}{4\cdot 6\cdots(4n)(4n+2)} x^{4n+2} =^{GeoGebra} sen(x^2) - x^2 + 1$ ❑

2) Aplique el método de Frobenius para la ecuación de Bessel de orden $\frac{1}{2}$

$$x^2 y'' + x\, y' + (x^2 - \frac{1}{4})\, y = 0$$

para obtener su solución general para $x > 0$,

$$y(x) = c_0\, \frac{cos(x)}{\sqrt{x}} + c_1\, \frac{sen(x)}{\sqrt{x}}$$

<u>Solución:</u>
Sea $y(x) = \sum_{n=0}^{\infty} c_n\, x^{n+r}$
Luego
$y'(x) = \sum_{n=0}^{\infty} (n+r)\, c_n\, x^{n+r-1}$
$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, c_n\, x^{n+r-2}$
Reemplazando en la ecuación diferencial

$$x^2y'' + x\,y' + (x^2 - \tfrac{1}{4})\,y = 0 \Rightarrow$$
$$\sum_{n=0}^{\infty}(n+r)(n+r-1)\,c_n\,x^{n+r} + \sum_{n=0}^{\infty}(n+r)\,c_n\,x^{n+r} + \sum_{n=0}^{\infty}c_n\,x^{n+r+2}$$
$$-\sum_{n=0}^{\infty}\tfrac{1}{4}c_n\,x^{n+r} = 0 \Rightarrow$$
$$r(r-1)c_0\,x^r + (r+1)r\,c_1x^{r+1} + \sum_{n=2}^{\infty}(n+r)(n+r-1)\,c_n\,x^{n+r} +$$
$$r\,c_0x^r + (r+1)\,c_1x^{r+1} + \sum_{n=2}^{\infty}(n+r)\,c_n\,x^{n+r} + \sum_{n=0}^{\infty}c_n\,x^{n+r+2}$$
$$-\tfrac{1}{4}c_0\,x^r - \tfrac{1}{4}c_1\,x^{r+1} - \sum_{n=2}^{\infty}\tfrac{1}{4}c_n\,x^{n+r} = 0 \Rightarrow$$
$$[r(r-1) + r - \tfrac{1}{4}]c_0\,x^r + [(r+1)\,r + r + 1 - \tfrac{1}{4}]c_1\,x^{r+1}$$
$$+\sum_{n=2}^{\infty}(n+r)(n+r-1)\,c_n\,x^{n+r} + \sum_{n=2}^{\infty}(n+r)\,c_n\,x^{n+r} + \sum_{n=0}^{\infty}c_n\,x^{n+r+2}$$
$$-\sum_{n=2}^{\infty}\tfrac{1}{4}c_n\,x^{n+r} = 0 \Rightarrow$$
$$[r(r-1) + r - \tfrac{1}{4}]c_0\,x^r + [(r+1)\,r + r + 1 - \tfrac{1}{4}]c_1\,x^{r+1}$$
$$+\sum_{n=0}^{\infty}(n+r+2)(n+r+1)\,c_{n+2}\,x^{n+r+2} + \sum_{n=0}^{\infty}(n+r+2)\,c_{n+2}\,x^{n+r+2}$$
$$+\sum_{n=0}^{\infty}c_n\,x^{n+r+2} - \sum_{n=0}^{\infty}\tfrac{1}{4}c_{n+2}\,x^{n+r+2} = 0 \Rightarrow$$
$$[r(r-1) + r - \tfrac{1}{4}]c_0\,x^r + [(r+1)\,r + r + 1 - \tfrac{1}{4}]c_1\,x^{r+1}$$
$$+\sum_{n=0}^{\infty}([(n+r+2)(n+r+1) + (n+r+2) - \tfrac{1}{4}]\,c_{n+2} + c_n)\,x^{n+r+2} = 0$$
La ecuación indicial se deduce de la relación
$$[r(r-1) + r - \tfrac{1}{4}]c_0 = 0 \Rightarrow^{c_0\neq 0} r(r-1) + r - \tfrac{1}{4} = 0 \Rightarrow r^2 - r + r - \tfrac{1}{4} = 0 \Rightarrow$$
$$r^2 = \tfrac{1}{4} \Rightarrow \begin{cases} r_1 = \tfrac{1}{2} \\ r_2 = -\tfrac{1}{2} \end{cases}$$
Por otro lado, observemos que si $r = -\frac{1}{2}$, entonces
$$[(r+1)\,r + r + 1 - \tfrac{1}{4}]\,c_1 = 0 \Rightarrow [r^2 + r + r + \tfrac{3}{4}]\,c_1 = 0 \Rightarrow$$
$[r^2 + 2r + \frac{3}{4}]\,c_1 = 0 \Rightarrow c_1 \neq 0,$ pues $(-\frac{1}{2})^2 + 2(-\frac{1}{2}) + \frac{3}{4} = \frac{1}{4} - 1 + \frac{3}{4} = 0$

Luego, probemos con la raíz menor, es decir, $r = -\frac{1}{2}$
$$[(n+r+2)(n+r+1) + (n+r+2) - \tfrac{1}{4}]\,c_{n+2} + c_n = 0 \Rightarrow$$
$$c_{n+2} = -\frac{c_n}{(n+r+2)(n+r+1)+(n+r+2)-\frac{1}{4}} \Rightarrow$$
$$c_{n+2} = -\frac{c_n}{(n+r+2)(n+r+2)-\frac{1}{4}} \Rightarrow$$
$$c_{n+2} = -\frac{c_n}{(n+r+2)^2-\frac{1}{4}} \Rightarrow c_{n+2} = -\frac{c_n}{(n-\frac{1}{2}+2)^2-\frac{1}{4}} \Rightarrow c_{n+2} = -\frac{c_n}{(n+\frac{3}{2})^2-\frac{1}{4}}$$
$$\Rightarrow c_{n+2} = -\frac{c_n}{n^2+3n+\frac{9}{4}-\frac{1}{4}} \Rightarrow c_{n+2} = -\frac{c_n}{n^2+3n+2} \Rightarrow c_{n+2} = -\frac{c_n}{(n+1)(n+2)}$$
$$n = 0 \Rightarrow c_2 = -\frac{c_0}{1\cdot 2}$$

$n=1 \Rightarrow c_3 = -\frac{c_1}{2\cdot 3}$
$n=2 \Rightarrow c_4 = -\frac{c_2}{3\cdot 4} = \frac{c_0}{1\cdot 2\cdot 3\cdot 4}$
$n=3 \Rightarrow c_5 = -\frac{c_3}{4\cdot 5} = \frac{c_1}{2\cdot 3\cdot 4\cdot 5}$
$n=4 \Rightarrow c_6 = -\frac{c_4}{5\cdot 6} = -\frac{c_0}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}$
$n=5 \Rightarrow c_7 = -\frac{c_5}{6\cdot 7} = -\frac{c_1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}$

Reemplazando en $y(x) = \sum_{n=0}^{\infty} c_n\, x^{n+r}$

$$y(x) = \sum_{n=0}^{\infty} c_n\, x^{n-\frac{1}{2}} = \sum_{n=0}^{\infty} c_n\, x^n\, x^{-\frac{1}{2}} = x^{-\frac{1}{2}}\sum_{n=0}^{\infty} c_n\, x^n = \frac{1}{\sqrt{x}}\sum_{n=0}^{\infty} c_n\, x^n$$

Ahora

$$\sum_{n=0}^{\infty} c_n\, x^n =$$
$$c_0 + c_1 x - \frac{c_0}{1\cdot 2}x^2 - \frac{c_1}{2\cdot 3}x^3 + \frac{c_0}{1\cdot 2\cdot 3\cdot 4}x^4 + \frac{c_1}{2\cdot 3\cdot 4\cdot 5}x^5 - \frac{c_0}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}x^6 - \frac{c_1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7 + \ldots$$
$$= c_0(1 - \frac{1}{1\cdot 2}x^2 + \frac{1}{1\cdot 2\cdot 3\cdot 4}x^4 - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}x^6 + \ldots) +$$
$$c_1(x - \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 4\cdot 5}x^5 - \frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7 + \ldots) =$$
$$c_0\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} + c_1\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = c_0\, cos(x) + c_1\, sen(x)$$

Finalmente,

$y(x) = \frac{1}{\sqrt{x}}\sum_{n=0}^{\infty} c_n\, x^n = c_0\,\frac{cos(x)}{\sqrt{x}} + c_1\,\frac{sen(x)}{\sqrt{x}}$, para $x > 0$ ❑

3) Muestre que la ecuación de Bessel de orden 1,

$$x^2 y'' + x\, y' + (x^2 - 1)\, y = 0$$

tiene exponentes $r_1 = 1$ y $r_2 = -1$ en $x = 0$.
Además, muestre que la serie de Frobenius correspondiente a $r_1 = 1$ es

$$J_1(x) = \frac{x}{2}\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{n!\,(n+1)!\,2^{2n}}$$

Obs.: $J_1(x)$ es la función de Bessel de primera clase y de orden 1.

Solución:

Sea $y(x) = \sum_{n=0}^{\infty} c_n\, x^{n+r}$

Luego

$$y'(x) = \sum_{n=0}^{\infty} (n+r)\, c_n\, x^{n+r-1}$$
$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, c_n\, x^{n+r-2}$$

Reemplazando en la ecuación diferencial

$x^2 y'' + x\, y' + (x^2 - 1)\, y = 0 \Rightarrow$

$\sum\limits_{n=0}^{\infty} (n+r)(n+r-1)\, c_n\, x^{n+r} + \sum\limits_{n=0}^{\infty} (n+r)\, c_n\, x^{n+r} + \sum\limits_{n=0}^{\infty} c_n\, x^{n+r+2}$

$-\sum\limits_{n=0}^{\infty} c_n\, x^{n+r} = 0 \Rightarrow$

$r(r-1)c_0\, x^r + (r+1)r\, c_1 x^{r+1} + \sum\limits_{n=2}^{\infty} (n+r)(n+r-1)\, c_n\, x^{n+r} +$

$r\, c_0 x^r + (r+1)\, c_1 x^{r+1} + \sum\limits_{n=2}^{\infty} (n+r)\, c_n\, x^{n+r} + \sum\limits_{n=0}^{\infty} c_n\, x^{n+r+2}$

$-\, c_0\, x^r - c_1\, x^{r+1} \; - \sum\limits_{n=2}^{\infty} c_n\, x^{n+r} = 0 \Rightarrow$

$[r(r-1) + r - 1]c_0\, x^r + [(r+1)\, r + r + 1 - 1]c_1\, x^{r+1}$

$+\sum\limits_{n=2}^{\infty} (n+r)(n+r-1)\, c_n\, x^{n+r} + \sum\limits_{n=2}^{\infty} (n+r)\, c_n\, x^{n+r} + \sum\limits_{n=0}^{\infty} c_n\, x^{n+r+2}$

$-\sum\limits_{n=2}^{\infty} c_n\, x^{n+r} = 0 \Rightarrow$

$[r(r-1) + r - 1]c_0\, x^r + [(r+1)\, r + r + 1 - 1]c_1\, x^{r+1}$

$+\sum\limits_{n=0}^{\infty} (n+r+2)(n+r+1)\, c_{n+2}\, x^{n+r+2} + \sum\limits_{n=0}^{\infty} (n+r+2)\, c_{n+2}\, x^{n+r+2}$

$+\sum\limits_{n=0}^{\infty} c_n\, x^{n+r+2} - \sum\limits_{n=0}^{\infty} c_{n+2}\, x^{n+r+2} = 0 \Rightarrow$

$[r(r-1) + r - 1]c_0\, x^r + [(r+1)\, r + r + 1 - 1]c_1\, x^{r+1}$

$+\sum\limits_{n=0}^{\infty} ([(n+r+2)(n+r+1) + (n+r+2) - 1]\, c_{n+2} + c_n)\, x^{n+r+2} = 0$

La ecuación indicial se deduce de la relación

$[r(r-1) + r - 1]c_0 = 0 \Rightarrow^{c_0 \neq 0} r(r-1) + r - 1 = 0 \Rightarrow r^2 - r + r - 1 = 0 \Rightarrow$

$r^2 = 1 \Rightarrow \begin{cases} r_1 = 1 \\ r_2 = -1 \end{cases}$

Notemos que para $r = 1$, se tiene que

$[(r+1)\, r + r + 1 - 1]c_1 = 0 \Rightarrow [r^2 + r + r]c_1 = 0 \Rightarrow [r^2 + 2r]c_1 = 0 \Rightarrow$

$(1+2)\, c_1 = 0 \Rightarrow 3c_1 = 0 \Rightarrow c_1 = 0$

Para $r_1 = 1$, se tiene

$[(n+1+2)(n+1+1) + (n+1+2) - 1]\, c_{n+2} + c_n = 0 \Rightarrow$

$c_{n+2} = -\frac{c_n}{(n+3)(n+2)+(n+2)} \Rightarrow c_{n+2} = -\frac{c_n}{(n+2)(n+4)}$

$n = 0 \Rightarrow c_2 = -\frac{c_0}{2 \cdot 4}$

$n = 1 \Rightarrow c_3 = -\frac{c_1}{3 \cdot 5} = 0$

$n = 2 \Rightarrow c_4 = -\frac{c_2}{4 \cdot 6} = \frac{c_0}{2 \cdot 4 \cdot 4 \cdot 6}$

$n = 3 \Rightarrow c_5 = -\frac{c_3}{5 \cdot 7} = \frac{c_1}{3 \cdot 5 \cdot 5 \cdot 7} = 0$

$n = 4 \Rightarrow c_6 = -\frac{c_4}{6 \cdot 8} = -\frac{c_0}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8}$

$n = 5 \Rightarrow c_7 = -\frac{c_5}{7 \cdot 9} = -\frac{c_1}{3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9} = 0$

$n = 6 \Rightarrow c_8 = -\frac{c_6}{8 \cdot 10} = \frac{c_0}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10}$

Reemplazando en $y(x) = \sum_{n=0}^{\infty} c_n\, x^{n+r}$

$y(x) = \sum_{n=0}^{\infty} c_n\, x^{n+1} = x \sum_{n=0}^{\infty} c_n\, x^n$

$\sum_{n=0}^{\infty} c_n\, x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + ...$

$$c_0 - \frac{c_0}{2{\cdot}4} x^2 + \frac{c_0}{2{\cdot}4{\cdot}4{\cdot}6} x^4 - \frac{c_0}{2{\cdot}4{\cdot}4{\cdot}6{\cdot}6{\cdot}8} x^6 + \frac{c_0}{2{\cdot}4{\cdot}4{\cdot}6{\cdot}6{\cdot}8{\cdot}8{\cdot}10} x^8 - ...$$
$$= c_0(1 - \frac{1}{2{\cdot}(2{\cdot}2)} x^2 + \frac{1}{2{\cdot}(2{\cdot}2){\cdot}(2{\cdot}2){\cdot}(2{\cdot}3)} x^4 - \frac{1}{2{\cdot}(2{\cdot}2){\cdot}(2{\cdot}2){\cdot}(2{\cdot}3){\cdot}(2{\cdot}3){\cdot}(2{\cdot}4)} x^6$$
$$+ \frac{c_0}{2{\cdot}(2{\cdot}2){\cdot}(2{\cdot}2){\cdot}(2{\cdot}3){\cdot}(2{\cdot}3){\cdot}(2{\cdot}4){\cdot}(2{\cdot}4){\cdot}(2{\cdot}5)} x^8 \; - ...) =$$
$$c_0(1 - \frac{1}{2{\cdot}2^2} x^2 + \frac{1}{2{\cdot}(2{\cdot}3){\cdot}2^4} x^4 - \frac{1}{(2{\cdot}3){\cdot}(2{\cdot}3{\cdot}4){\cdot}2^6} x^6 + \frac{c_0}{(2{\cdot}3{\cdot}4){\cdot}(2{\cdot}3{\cdot}4{\cdot}5){\cdot}2^8} x^8 \; - ...) =$$
$$c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+1)!\,2^{2n}}\, x^{2n}$$

Finalmente,

$y(x) = x\, c_0 \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{n!\,(n+1)!\,2^{2n}} = c\, \frac{x}{2} \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{n!\,(n+1)!\,2^{2n}} = c\, J_1(x)$, con $c_0 = \frac{c}{2}$ ❑