

PAUTA TEST N° 2 CÁLCULO II
INGENIERÍA AMBIENTAL

NOMBRE : _____

TIEMPO MÁXIMO : 60 MINUTOS

FECHA : Mi 07/10/26

1) Obtenga el valor exacto de $\int_0^1 \frac{\sqrt{x+3}}{x+2} dx$

(30 puntos)

Solución:

Resolvamos la integral indefinida $\int \frac{\sqrt{x+3}}{x+2} dx$

Sea $z^2 = x + 3$

$$z^2 = x + 3 \Rightarrow 2z dz = dx$$

$$z^2 = x + 3 \Rightarrow x = z^2 - 3 \Rightarrow x + 2 = z^2 - 3 + 2 \Rightarrow x + 2 = z^2 - 1$$

$$\int \frac{\sqrt{x+3}}{x+2} dx = \int \frac{z}{z^2-1} 2z dz = 2 \int \frac{z^2}{z^2-1} dz$$

Debemos dividir:

$$z^2 : z^2 - 1 = 1$$

$$\begin{array}{r} (-) z^2 - (+) 1 \\ \hline \end{array}$$

$$1$$

Luego: $\frac{z^2}{z^2-1} = 1 + \frac{1}{z^2-1}$

$$\int \frac{z^2}{z^2-1} dz = \int 1 + \frac{1}{z^2-1} dz = \int dz + \int \frac{1}{z^2-1} dz = z + \int \frac{1}{z^2-1} dz$$

Calculemos la integral $\int \frac{1}{z^2-1} dz$ usando descomposición en suma de fracciones parciales

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{A}{z-1} + \frac{B}{z+1} \Rightarrow 1 = A(z+1) + B(z-1)$$

$$z = -1 : 1 = -2B \Rightarrow B = -\frac{1}{2}$$

$$z = 1 : 1 = 2A \Rightarrow A = \frac{1}{2}$$

De lo anterior, se tiene que:

$$\frac{1}{z^2-1} = \frac{1}{2} \frac{1}{z-1} - \frac{1}{2} \frac{1}{z+1} \Rightarrow \int \frac{1}{z^2-1} dz = \frac{1}{2} \int \frac{1}{z-1} dz - \frac{1}{2} \int \frac{1}{z+1} dz \Rightarrow$$

$$\int \frac{1}{z^2-1} dz = \frac{1}{2} \ln|z-1| - \frac{1}{2} \ln|z+1|$$

Reuniendo los resultados anteriores tenemos que:

$$\int \frac{\sqrt{x+3}}{x+2} dx = 2 \int \frac{z^2}{z^2-1} dz = 2z + \ln|z-1| - \ln|z+1| =$$

$$2\sqrt{x+3} + \ln|\sqrt{x+3}-1| - \ln|\sqrt{x+3}+1|$$

Finalmente,

$$\int_0^1 \frac{\sqrt{x+3}}{x+2} dx = \left[2\sqrt{x+3} + \ln|\sqrt{x+3}-1| - \ln|\sqrt{x+3}+1| \right]_0^1 =$$

$$2\sqrt{4} + \ln|\sqrt{4}-1| - \ln|\sqrt{4}+1| - 2\sqrt{3} - \ln|\sqrt{3}-1| + \ln|\sqrt{3}+1| =$$

$$4 + \ln(1) - \ln(3) - 2\sqrt{3} - \ln|\sqrt{3}-1| + \ln|\sqrt{3}+1| =$$

$$4 - \ln(3) - 2\sqrt{3} - \ln|\sqrt{3}-1| + \ln|\sqrt{3}+1| \quad \square$$

2) Calcule $\frac{1}{2} \int \frac{z-2}{z^2-z+1} dz$

(30 puntos)

Solución:

Completemos cuadrados en la expresión $z^2 - z + 1$

$$z^2 - z + 1 = (z^2 - z) + 1 = (z - \frac{1}{2})^2 + 1 - \frac{1}{4} = (z - \frac{1}{2})^2 + \frac{3}{4}$$

Consideremos la sustitución trigonométrica $z - \frac{1}{2} = \frac{\sqrt{3}}{2} \operatorname{tg}(\alpha)$

$$(z - \frac{1}{2})^2 + \frac{3}{4} = \frac{3}{4} \operatorname{tg}^2(\alpha) + \frac{3}{4} = \frac{3}{4} (\operatorname{tg}^2(\alpha) + 1) = \frac{3}{4} \sec^2(\alpha)$$

$$z - \frac{1}{2} = \frac{\sqrt{3}}{2} \operatorname{tg}(\alpha) \Rightarrow z = \frac{\sqrt{3}}{2} \operatorname{tg}(\alpha) + \frac{1}{2} \Rightarrow z - 2 = \frac{\sqrt{3}}{2} \operatorname{tg}(\alpha) + \frac{1}{2} - 2 \Rightarrow$$

$$z - 2 = \frac{\sqrt{3}}{2} \operatorname{tg}(\alpha) - \frac{3}{2} \Rightarrow dz = \frac{\sqrt{3}}{2} \sec^2(\alpha) d\alpha$$

$$\frac{1}{2} \int \frac{z-2}{z^2-z+1} dz = \frac{1}{2} \int \frac{\frac{\sqrt{3}}{2} \operatorname{tg}(\alpha) - \frac{3}{2}}{\frac{3}{4} \sec^2(\alpha)} \frac{\sqrt{3}}{2} \sec^2(\alpha) d\alpha =$$

$$\frac{1}{2} \int \frac{\frac{1}{2}(\sqrt{3} \operatorname{tg}(\alpha)-3)}{\frac{3}{4} \sec^2(\alpha)} \frac{\sqrt{3}}{2} \sec^2(\alpha) d\alpha = \frac{1}{2} \frac{2}{3} \frac{\sqrt{3}}{2} \int (\sqrt{3} \operatorname{tg}(\alpha) - 3) d\alpha =$$

$$\frac{\sqrt{3}}{6} \int (\sqrt{3} \operatorname{tg}(\alpha) - 3) d\alpha = \frac{1}{2} \int \operatorname{tg}(\alpha) d\alpha - \frac{\sqrt{3}}{2} \int d\alpha =$$

$$- \frac{1}{2} \ln |\cos(\alpha)| - \frac{\sqrt{3}}{2} \alpha + c = \frac{1}{2} \ln |\cos^{-1}(\alpha)| - \frac{\sqrt{3}}{2} \alpha + c =$$

$$\frac{1}{2} \ln |\sec(\alpha)| - \frac{\sqrt{3}}{2} \alpha + c =$$

Volvamos a la variable original

$$z - \frac{1}{2} = \frac{\sqrt{3}}{2} \operatorname{tg}(\alpha) \Rightarrow \operatorname{tg}(\alpha) = \frac{2}{\sqrt{3}} \left(z - \frac{1}{2}\right) \Rightarrow \alpha = \operatorname{Arctg} \left(\frac{2}{\sqrt{3}} \left(z - \frac{1}{2}\right) \right)$$

Ahora usemos la identidad $\operatorname{tg}^2(\alpha) + 1 = \sec^2(\alpha)$, **para obtener** $\sec(\alpha)$ **en términos de** $\operatorname{tg}(\alpha)$

$$\operatorname{tg}^2(\alpha) + 1 = \sec^2(\alpha) \Rightarrow \sec(\alpha) = \sqrt{\operatorname{tg}^2(\alpha) + 1} \Rightarrow \sec(\alpha) = \sqrt{\frac{4}{3} \left(z - \frac{1}{2}\right)^2 + 1}$$

Finalmente,

$$\frac{1}{2} \int \frac{z-2}{z^2-z+1} dz = \frac{1}{2} \ln \left| \sqrt{\frac{4}{3} \left(z - \frac{1}{2}\right)^2 + 1} \right| - \frac{\sqrt{3}}{2} \operatorname{Arctg} \left(\frac{2}{\sqrt{3}} \left(z - \frac{1}{2}\right) \right) + c \quad \square$$

Obs.: En el test no se debe usar calculadora ni celular.