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PAUTA CERTAMEN N° 2 CÁLCULO II
INGENIERÍA AMBIENTAL

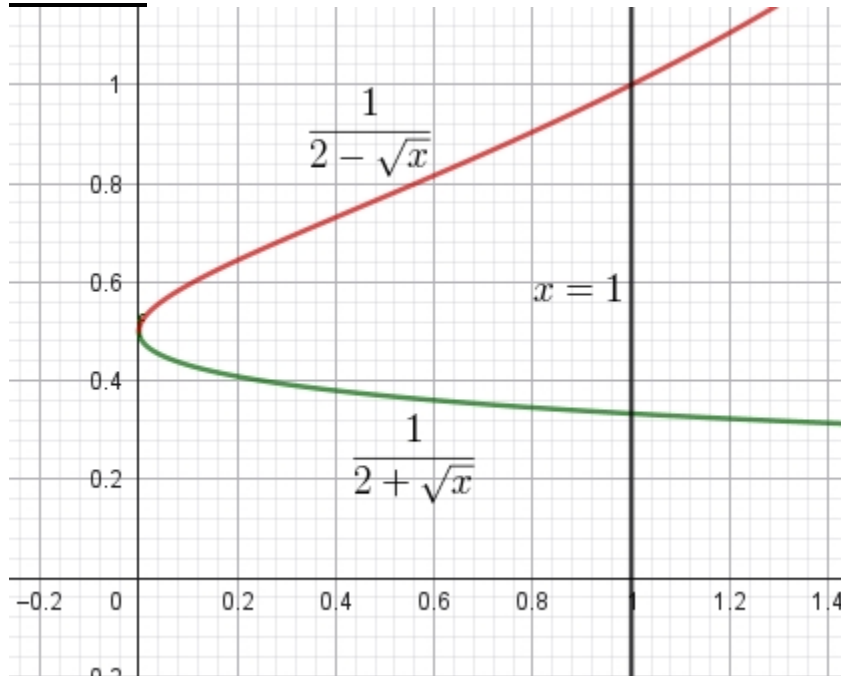
NOMBRE : _____

TIEMPO MÁXIMO : 1 HORA 30 MINUTOS

FECHA : Vi 24/10/25

- 1) Obtenga el área de la región acotada por $y = \frac{1}{2+\sqrt{x}}$, $y = \frac{1}{2-\sqrt{x}}$ y $x = 1$
(15 puntos)

Solución:



$$A = \int_0^1 \left(\frac{1}{2-\sqrt{x}} - \frac{1}{2+\sqrt{x}} \right) dx = \int_0^1 \frac{2+\sqrt{x}-2+\sqrt{x}}{4-x} dx = \int_0^1 \frac{2\sqrt{x}}{4-x} dx = 2 \int_0^1 \frac{\sqrt{x}}{4-x} dx$$

Resolvamos la integral indefinida $\int \frac{\sqrt{x}}{4-x} dx$

$$z^2 = x \Rightarrow 2z dz = dx$$

$$z^2 = x \Rightarrow z = \sqrt{x}$$

$$\int \frac{\sqrt{x}}{4-x} dx = \int \frac{z}{4-z^2} 2z dz = 2 \int \frac{z^2}{4-z^2} dz = -2 \int \frac{z^2}{z^2-4} dz$$

$$z^2 : z^2 - 4 = 1$$

$$(-)z^2 - (+)4$$

$$4$$

$$\frac{z^2}{z^2-4} = 1 + \frac{4}{z^2-4}$$

$$\int \frac{z^2}{z^2-4} dz = \int dz + \int \frac{4}{z^2-4} dz = z + 4 \int \frac{1}{z^2-4} dz$$

$$\frac{1}{z^2-4} = \frac{1}{(z-2)(z+2)} = \frac{A}{z-2} + \frac{B}{z+2} \Rightarrow 1 = A(z+2) + B(z-2)$$

$$z = 2 : 1 = 4A \Rightarrow A = \frac{1}{4}$$

$$z = -2 : 1 = -4B \Rightarrow B = -\frac{1}{4}$$

$$4 \int \frac{1}{z^2-4} dz = \int \frac{1}{z-2} dz - \int \frac{1}{z+2} dz = \ln|z-2| - \ln|z+2|$$

$$\int \frac{z^2}{z^2-4} dz = z + 4 \int \frac{1}{z^2-4} dz = z + \ln|z-2| - \ln|z+2|$$

$$\int \frac{\sqrt{x}}{4-x} dx = -2 \int \frac{z^2}{z^2-4} dz = -2z - 2\ln|z-2| + 2\ln|z+2| =$$

$$-2\sqrt{x} - 2\ln|\sqrt{x}-2| + 2\ln|\sqrt{x}+2|$$

$$A = 2 \int_0^1 \frac{\sqrt{x}}{4-x} dx = \left(-4\sqrt{x} - 4\ln|\sqrt{x}-2| + 4\ln|\sqrt{x}+2| \right) \Big|_0^1 =$$

$$-4 - 4\ln|-1| + 4\ln|3| + 0 + 4\ln|-2| - 4\ln|2| = -4 + 4\ln(3)$$

Finalmente, el área es $A = 4\ln(3) - 4 \approx 0.39$ $\square$

2) Calcule $\int_1^\infty \frac{e^x}{1+e^x} dx$

(15 puntos)

Solución:

$$\int_1^\infty \frac{e^x}{1+e^x} dx = \lim_{p \rightarrow \infty} \int_1^p \frac{e^x}{1+e^x} dx$$

Resolvamos la integral indefinida $\int \frac{e^x}{1+e^x} dx$

Sea $B = 1 + e^x$

$$B = 1 + e^x \Rightarrow dB = e^x dx$$

$$\int \frac{e^x}{1+e^x} dx = \int \frac{1}{B} dB = \ln(B) = \ln(1 + e^x)$$

$$\int_1^p \frac{e^x}{1+e^x} dx = \ln(1 + e^p) - \ln(1 + e)$$

$$\int_1^{\infty} \frac{e^x}{1+e^x} dx = \lim_{p \rightarrow \infty} \int_1^p \frac{e^x}{1+e^x} dx = \lim_{p \rightarrow \infty} [\ln(1+e^p) - \ln(1+e)] = \infty$$

Esto muestra que la integral diverge. $\square$

3) a) Escriba la integral para calcular la longitud de arco de $x = \ln(t)$, $y = e^{-t}$, $1 \leq t \leq 2$

b) Calcule una aproximación de la integral anterior, usando el método de Simpson con $n = 6$

(15 puntos)

Solución:

a) Recordemos que la longitud de arco L está dada por

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$\frac{dx}{dt} = \frac{1}{t}$$

$$\frac{dy}{dt} = -e^{-t}$$

$$L = \int_1^2 \sqrt{\left(\frac{1}{t}\right)^2 + \left(-e^{-t}\right)^2} dt = \int_1^2 \sqrt{\frac{1}{t^2} + e^{-2t}} dt \quad \square$$

b) Si $n = 6$, entonces $h = \frac{2-1}{6} = \frac{1}{6}$

Además,

$$t_0 = 1, t_1 = \frac{7}{6}, t_2 = \frac{8}{6} = \frac{4}{3}, t_3 = \frac{9}{6} = \frac{3}{2}, t_4 = \frac{10}{6} = \frac{5}{3}, t_5 = \frac{11}{6}, t_6 = \frac{12}{6} = 2$$

$$f(t) = \sqrt{\frac{1}{t^2} + e^{-2t}}$$

$$f(t_0) = f(1) \approx 1.06552113223$$

$$f(t_1) = f\left(\frac{7}{6}\right) \approx 0.911957151085$$

$$f(t_2) = f\left(\frac{4}{3}\right) \approx 0.794973868264$$

$$f(t_3) = f\left(\frac{3}{2}\right) \approx 0.703016011775$$

$$f(t_4) = f\left(\frac{5}{3}\right) \approx 0.629026226279$$

$$f(t_5) = f\left(\frac{11}{6}\right) \approx 0.568403197003$$

$$f(t_6) = f(2) \approx 0.517991929366$$

$$\int_1^2 \sqrt{\frac{1}{t^2} + e^{-2t}} dt \approx$$

$$\frac{h}{3} \left[f(t_0) + 4f(t_1) + 2f(t_2) + 4f(t_3) + 2f(t_4) + 4f(t_5) + f(t_6) \right] =$$

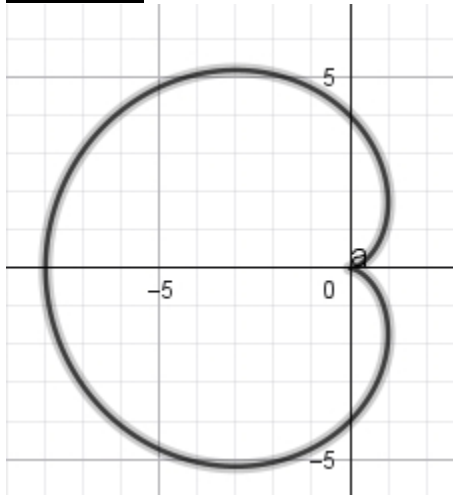
$$\frac{1}{18} \left[1.06552113223 + 4(0.911957151085) + 2(0.794973868264) + 4(0.703016011775) + 2(0.629026226279) + 4(0.568403197003) + 0.517991929366 \right] = \frac{1}{18}(13.165018690142) = 0.731389927230$$

Finalmente, $\int_1^2 \sqrt{\frac{1}{t^2} + e^{-2t}} dt \approx 0.731389927230$ $\square$

4) Obtenga el área que encierra la curva $r = 4(1 - \cos(\alpha))$

(15 puntos)

Solución:



$$\begin{aligned} A &= \frac{1}{2} \int_{\beta}^{\delta} r^2(\alpha) d\alpha = \frac{1}{2} \int_0^{2\pi} 16(1 - \cos(\alpha))^2 d\alpha = \\ &= 8 \int_0^{2\pi} [1 - 2\cos(\alpha) + \cos^2(\alpha)] d\alpha = \\ &= 8 \int_0^{2\pi} d\alpha - 16 \int_0^{2\pi} \cos(\alpha) d\alpha + 8 \int_0^{2\pi} \cos^2(\alpha) d\alpha = \\ &= 8\alpha \Big|_0^{2\pi} - 16\sin(\alpha) \Big|_0^{2\pi} + 8 \int_0^{2\pi} \frac{1}{2}(1 + \cos(2\alpha)) d\alpha = \\ &= 16\pi + 4 \int_0^{2\pi} d\alpha + 4 \int_0^{2\pi} \cos(2\alpha) d\alpha = 16\pi + 4\alpha \Big|_0^{2\pi} + \frac{4}{2} \sin(2\alpha) \Big|_0^{2\pi} \\ &= 16\pi + 8\pi + 2\sin(2\alpha) \Big|_0^{2\pi} = 16\pi + 8\pi = 24\pi \quad \square \end{aligned}$$