

**PAUTA CERTAMEN N° 1 CÁLCULO AVANZADO
INGENIERÍA AMBIENTAL**

NOMBRE : _____

TIEMPO MÁXIMO : 1 HORA 40 MINUTOS

FECHA : Lu 13/04/26

1) Resuelva la integral $\int \frac{dt}{a^2+b^2t}$; $a, b \in \mathbb{R}$

(15 puntos)

Solución:

Sea $v = a^2 + b^2t$

$$v = a^2 + b^2t \Rightarrow dv = b^2 dt \Rightarrow b \neq 0 \frac{1}{b^2} dv = dt$$

$$\int \frac{dt}{a^2+b^2t} = \frac{1}{b^2} \int \frac{dv}{v} = \frac{1}{b^2} \ln|v| + c_1 = \frac{1}{b^2} \ln|a^2 + b^2t| + c, \text{ para } b \neq 0$$

Consideremos ahora el caso cuando $b = 0$ y $a \neq 0$

$$\int \frac{dt}{a^2+b^2t} = \int \frac{dt}{a^2} = \frac{1}{a^2} \int dt = \frac{1}{a^2} t + c_2 \quad \square$$

2) Muestre que : $\int_0^1 t \cdot \text{Arcsen}(t) dt = \frac{\pi}{8}$

(15 puntos)

Solución:

Resolvamos la integral indefinida $\int t \cdot \text{Arcsen}(t) dt$ usando integración por partes

$$p' = t \Rightarrow p = \frac{1}{2} t^2$$

$$q = \text{Arcsen}(t) \Rightarrow q' = \frac{1}{\sqrt{1-t^2}}$$

$$\int t \cdot \text{Arcsen}(t) dt = \frac{1}{2} t^2 \text{Arcsen}(t) - \frac{1}{2} \int \frac{t^2}{\sqrt{1-t^2}} dt$$

Resolvamos ahora la integral $\int \frac{t^2}{\sqrt{1-t^2}} dt$ usando sustitución trigonométrica

$$t = \text{sen}(\alpha) \Rightarrow dt = \cos(\alpha) d\alpha$$

$$\sqrt{1-t^2} = \sqrt{1-\text{sen}^2(\alpha)} = \sqrt{\cos^2(\alpha)} = \cos(\alpha)$$

$$t^2 = \text{sen}^2(\alpha)$$

$$\int \frac{t^2}{\sqrt{1-t^2}} dt = \int \frac{\text{sen}^2(\alpha)}{\cos(\alpha)} \cos(\alpha) d\alpha = \int \text{sen}^2(\alpha) d\alpha = \frac{1}{2} \int (1 - \cos(2\alpha)) d\alpha = \frac{1}{2} \int d\alpha - \frac{1}{2} \int \cos(2\alpha) d\alpha = \frac{1}{2} \alpha - \frac{1}{2} \frac{\text{sen}(2\alpha)}{2} + c = \frac{1}{2} \alpha - \frac{1}{4} \text{sen}(2\alpha)$$

Volvamos a la variable original t

$$t = \text{sen}(\alpha) \Rightarrow \alpha = \text{Arcsen}(t)$$

$$\text{sen}(2\alpha) = 2 \text{sen}(\alpha) \cos(\alpha) = 2 \text{sen}(\alpha) \sqrt{1 - \text{sen}^2(\alpha)} = 2 t \sqrt{1 - t^2}$$

$$\int \frac{t^2}{\sqrt{1-t^2}} dt = \frac{1}{2} \text{Arcsen}(t) - \frac{1}{4} 2 t \sqrt{1-t^2} + c = \frac{1}{2} \text{Arcsen}(t) - \frac{1}{2} t \sqrt{1-t^2}$$

Luego

$$\int t \cdot \text{Arcsen}(t) dt = \frac{1}{2} t^2 \text{Arcsen}(t) - \frac{1}{2} \int \frac{t^2}{\sqrt{1-t^2}} dt = \frac{1}{2} t^2 \text{Arcsen}(t) - \frac{1}{4} \text{Arcsen}(t) + \frac{1}{4} t \sqrt{1-t^2} + c$$

Volviendo a la integral definida, se tiene que

$$\int_0^1 t \cdot \text{Arcsen}(t) dt = \left[\frac{1}{2} t^2 \text{Arcsen}(t) - \frac{1}{4} \text{Arcsen}(t) + \frac{1}{4} t \sqrt{1-t^2} \right]_0^1 = \frac{1}{2} \text{Arcsen}(1) - \frac{1}{4} \text{Arcsen}(1) + \frac{1}{4} \text{Arcsen}(0) = \frac{1}{4} \text{Arcsen}(1) = \frac{1}{4} \frac{\pi}{2} = \frac{\pi}{8} \quad \square$$

3) Resuelva $\int \cos^3(q) dq$

(15 puntos)

Solución:

$$\int \cos^3(q) dq = \int \cos^2(q) \cos(q) dq = \int [1 - \text{sen}^2(q)] \cos(q) dq =$$

$$\int \cos(q) dq - \int \text{sen}^2(q) \cos(q) dq = \text{sen}(q) - \int \text{sen}^2(q) \cos(q) dq$$

Para calcular la integral $\int \text{sen}^2(q) \cos(q) dq$ usamos la sustitución $P = \text{sen}(q)$

$$P = \text{sen}(q) \Rightarrow dP = \cos(q) dq$$

Luego

$$\int \text{sen}^2(q) \cos(q) dq = \int P^2 dP = \frac{1}{3} P^3 - c = \frac{1}{3} \text{sen}^3(q) - c$$

$$\text{Finalmente, } \int \cos^3(q) dq = \text{sen}(q) - \frac{1}{3} \text{sen}^3(q) + c \quad \square$$

4) Resuelva la EDO $y' = \frac{1}{\sqrt{1+e^x}}$

(15 puntos)

Solución:

$$y' = \frac{1}{\sqrt{1+e^x}} \Rightarrow \int y' dx = \int \frac{1}{\sqrt{1+e^x}} dx \Rightarrow y(x) = \int \frac{1}{\sqrt{1+e^x}} dx$$

Sea $z^2 = 1 + e^x$

$$z^2 = 1 + e^x \Rightarrow z = \sqrt{1 + e^x}$$

$$z^2 = 1 + e^x \Rightarrow e^x = z^2 - 1$$

$$z^2 = 1 + e^x \Rightarrow 2z dz = e^x dx \Rightarrow 2z dz = (z^2 - 1) dx \Rightarrow dx = \frac{2z}{z^2 - 1} dz$$

$$\int \frac{1}{\sqrt{1+e^x}} dx = \int \frac{1}{z} \frac{2z}{z^2 - 1} dz = 2 \int \frac{1}{z^2 - 1} dz$$

Para calcular $\int \frac{1}{z^2 - 1} dz$ usemos factorización en suma de fracciones parciales

$$\frac{1}{z^2 - 1} = \frac{1}{(z-1)(z+1)} = \frac{A}{z-1} + \frac{B}{z+1} = \frac{A(z+1)+B(z-1)}{(z-1)(z+1)} \Rightarrow 1 = A(z+1) + B(z-1)$$

$$z = 1 : 1 = 2A \Rightarrow A = \frac{1}{2}$$

$$z = -1 : 1 = -2B \Rightarrow B = -\frac{1}{2}$$

$$\text{Luego } \frac{1}{z^2 - 1} = \frac{1}{2} \frac{1}{z-1} - \frac{1}{2} \frac{1}{z+1}$$

$$\int \frac{1}{z^2 - 1} dz = \frac{1}{2} \int \frac{1}{z-1} dz - \frac{1}{2} \int \frac{1}{z+1} dz = \frac{1}{2} \ln|z-1| - \frac{1}{2} \ln|z+1| + c$$

Finalmente,

$$\int \frac{1}{\sqrt{1+e^x}} dx = 2 \int \frac{1}{z^2 - 1} dz = \ln|z-1| - \ln|z+1| + c =$$

$$\ln|\sqrt{1+e^x} - 1| - \ln|\sqrt{1+e^x} + 1| + c \quad \square$$