Numerical stability in the solution of the flow in an hydrocyclone

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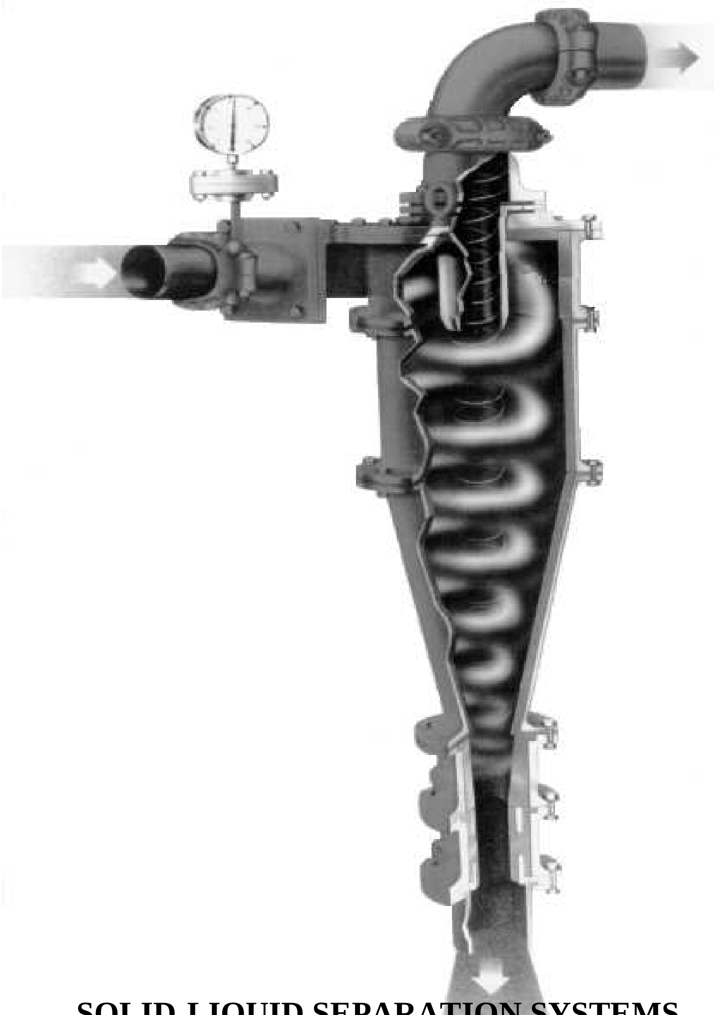
SOLID-LIQUID SEPARATION SYSTEMS

The hydrocyclone



Formulation: streamfunction - vorticity

- Velocity $\mathbf{u} \equiv (u, v, w)$
- Vorticity $\mathbf{w} = \nabla \times \mathbf{v} \equiv (\xi, \eta, \zeta)$
- Streamfunction $u = -\frac{1}{r}\frac{\partial \psi}{\partial z} \qquad w = \frac{1}{r}\frac{\partial \psi}{\partial r}$
- Circulation $\Gamma = vr$

$$\nabla \cdot \mathbf{v} = 0$$

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{v}$$

$$\frac{D\mathbf{w}}{Dt} = \nabla \mathbf{v} \cdot \mathbf{w} + \nu \nabla^2 \mathbf{w}$$

Formulation streamfunction – vorticity...

$$\frac{\partial \eta}{\partial t} = \frac{1}{r^3}\frac{\partial \Gamma^2}{\partial z} + \frac{u\eta}{r} - u\frac{\partial \eta}{\partial r} - w\frac{\partial \eta}{\partial z} + \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\eta)\right) + \frac{\partial^2 \eta}{\partial z^2}\right]$$

$$\frac{\partial \Gamma}{\partial t} = -u\frac{\partial \Gamma}{\partial r} - w\frac{\partial \Gamma}{\partial z} + \nu\left[r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Gamma}{\partial r}\right) + \frac{\partial^2 \Gamma}{\partial z^2}\right]$$

$$\eta = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) - \frac{1}{r}\frac{\partial^2 \psi}{\partial z^2}$$

If $\beta = \eta/r$

$$\frac{\partial \beta}{\partial t} + u\frac{\partial \beta}{\partial r} + w\frac{\partial \beta}{\partial z} = \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \beta}{\partial r}\right) + \frac{2}{r}\frac{\partial \beta}{\partial r} + \frac{\partial^2 \beta}{\partial z^2}\right] + \frac{\partial}{\partial z}\left(\frac{\Gamma^2}{r^4}\right)$$

Formulation streamfunction – vorticity...

Advection-diffusion equation for a tracer c

$$\frac{\partial c}{\partial t} + \mathbf{v} \cdot \nabla c = \nu \nabla^2 c + f$$

If

$$\frac{\nu}{|\mathbf{v}| h} \ll O(1)$$

the standard numerical methods loose stability

Stabilizated solution

- Solution using Finite Difference (Hsie, 1988)
- Our solution: use Finite Element
  - Boundary conditions are more direct
  - Use of complex and nonstructured meshes
  - A extension of USFEM method of Franca/Valentin (2000) for advective diffusive equations, using cylindrical coordinates

Stabilizated solution...

Consider the differential equation

$$\mathcal{L}u = f$$

we to replace the variational for $(\mathcal{L}u, v) = (f, v)$
by the stabilized formulation

$$(\mathcal{L}u, v) - \sum_{K \in C_h} (\mathcal{L}u, \tau \mathcal{L}^{\dagger} v)_K = (f, v) - \sum_{K \in C_h} (f, \tau \mathcal{L}^{\dagger} v)_K$$

Stabilizated solution...

where $\mathcal{L}^{\dagger}$ is a discrete adjunct operator to $\mathcal{L}$.

$$(\mathcal{L}u, v) = \int_{\Omega} \mathcal{L}(u) v d\Omega = \int_{\Omega} u \mathcal{L}^{\dagger}(v) d\Omega = (u, \mathcal{L}^{\dagger} v)$$

τ is a parameter depending on the element K of C_h

the terms that produce the stabilization contains functions that belong to the discrete space $V_h \subset H_0^1(\Omega)$.

Stabilizated solution...

- Circulation equation

$$\dot{\Gamma} + \mathbf{a} \cdot \nabla \Gamma - \nu \left(\nabla^2 \Gamma - \frac{1}{r} \frac{\partial \Gamma}{\partial r} \right) = 0 \text{ in } \Omega \qquad \mathbf{a} \equiv (u, w)$$

Discrete Galerkin formulation: find $u_h \in V_h$ such that

$$(\dot{u}_h, v_h) + (\mathbf{a} \cdot \nabla u_h, v_h) - \nu(\nabla^2 u_h, v_h) + \nu(\frac{1}{r} \frac{\partial u_h}{\partial r}, v_h) = 0 \ , \forall v_h \in V_h$$

USFEM formulation: $(\dot{u}_h, v_h) + B(u_h, v_h) = 0 \ , \forall v_h \in V_h$

$$B(u,v) = (\mathbf{a} \cdot \nabla u, v) + \nu(\nabla u, \nabla v) + 2\nu(\frac{1}{r} \frac{\partial u}{\partial r}, v) +$$
$$\sum_{K \in C_h} \left(\mathbf{a} \cdot \nabla u - \nu \triangle u + \frac{\nu}{r} \frac{\partial u}{\partial r}, \tau_K \left(\mathbf{a} \cdot \nabla v + \nu \triangle v + \frac{3\nu}{r} \frac{\partial v}{\partial r} + \frac{a_r v}{r} \right) \right)_K$$

Stabilizated solution...

- Reduced Vorticity equation

$$\dot{\beta} + \mathbf{a}\cdot\nabla\beta - \nu\left(\nabla^2\beta + \frac{3}{r}\frac{\partial\beta}{\partial r}\right) = f^{(\beta)} \text{ in } \Omega \qquad \mathbf{a} \equiv (u, w)$$

Discrete Galerkin formulation: find $u_h \in V_h$ such that

$$(\dot{u}_h, v_h) + (\mathbf{a}\cdot\nabla u_h, v_h) - \nu(\nabla^2 u_h, v_h) - 3\nu\left(\frac{1}{r}\frac{\partial u_h}{\partial r}, v_h\right) = (f^{(\beta)}, v_h) \;, \forall v_h \in V_h$$

USFEM formulation: $(\dot{u}_h, v_h) + B(u_h, v_h) = F(v_h) \;, \forall v_h \in V_h$

$$B(u,v) = (\mathbf{a}\cdot\nabla u, v) + \nu(\nabla u, \nabla v) - 2\nu\left(\frac{1}{r}\frac{\partial u}{\partial r}, v\right) +$$
$$\sum_{K\in C_h}\left(\mathbf{a}\cdot\nabla u - \nu\Delta u - \frac{3\nu}{r}\frac{\partial u}{\partial r}, \tau_K\left(\mathbf{a}\cdot\nabla v + \nu\Delta v - \frac{\nu}{r}\frac{\partial v}{\partial r} + \frac{a_r v}{r}\right)\right)_K$$

$$F(v) = (f^{(\beta)}, v) + \sum_{K\in C_h}\left(f^{(\beta)}, \tau_K\left(\mathbf{a}\cdot\nabla v + \nu\Delta v - \frac{\nu}{r}\frac{\partial v}{\partial r} + \frac{a_r v}{r}\right)\right)_K$$

Stabilizated solution...

- Streamfunction equation

$$\beta r^2 = -\frac{\partial^2 \psi}{\partial r^2} - \frac{\partial^2 \psi}{\partial z^2} + \frac{1}{r}\frac{\partial \psi}{\partial r}$$

Standard Galerkin formulation:

$$(\beta r^2, v) = (\nabla \psi, \nabla v) + 2\left(\frac{1}{r}\frac{\partial \psi}{\partial r}, v\right)$$

$$\nabla \equiv \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial z}\right) \qquad \nabla^2 \equiv \left(\frac{\partial^2}{\partial r^2}, \frac{\partial^2}{\partial z^2}\right)$$

Stabilizated solution...

$$\tau_K = \frac{h_K^2}{6\nu \cdot max\{Pe_K(\mathbf{x}), 1\}}$$

where

$$Pe_K(\mathbf{x}) = \frac{3|\mathbf{a}(\mathbf{x})|_2 h_K}{\nu}$$

$$|\mathbf{a}(\mathbf{x})| = \left(\Sigma_{i=1}^{N} |a_i(\mathbf{x})|^2\right)^{\frac{1}{2}}$$

The formula for τ_K has a form that is suggested by static condensation, as explained in [7].

Integration in time

given

$$[A]\dot{\phi} + [B]\phi = [F]$$

the solution is

$$\phi^{t+\Delta t} = ([A] + \Delta t[B])^{-1}([A]\phi^t + [F]\Delta t)$$

Boundary conditions

- Walls $\psi_o \quad , \quad \psi_u$
- Inlet $\psi_o \leq \psi \leq \psi_u$
- Outlets $\dfrac{\partial \psi}{\partial n} = 0$
- Free surface $f \equiv Q_o / Q_u \qquad f = \dfrac{\psi_o - \psi_{fs}}{\psi_{fs} - \psi_u}$

Boundary conditions...

- Dirichlet boundaries

$$[A]\{\psi\} = \{B\}$$

If ψ is know over m nodes belong to the contour Γ, $\tilde{\Omega} \equiv \Omega - \Gamma$, we can split the vectorial space for ψ in two disjoint subspaces ψ_Γ and $\psi_{\tilde{\Omega}}$ in the form

$$\begin{bmatrix} A_{\Gamma\Gamma} & A_{\Gamma\tilde{\Omega}} \\ A_{\tilde{\Omega}\Gamma} & A_{\tilde{\Omega}\tilde{\Omega}} \end{bmatrix} \begin{Bmatrix} \psi_\Gamma \\ \psi_{\tilde{\Omega}} \end{Bmatrix} = \begin{Bmatrix} B_\Gamma \\ B_{\tilde{\Omega}} \end{Bmatrix}$$

the solution for $\psi_{\tilde{\Omega}}$ is

$$\psi_{\tilde{\Omega}} = A^{-1}_{\tilde{\Omega}\tilde{\Omega}}(B_{\tilde{\Omega}} - A_{\tilde{\Omega}\Gamma}\psi_{\tilde{\Omega}})$$

Results

- Turbulence: Prandlt model

$$\mu_t = K(R_c, \rho_m, \lambda(z,r), \mu_0)\left(\left|\frac{1}{r}\frac{\partial\Gamma}{\partial r} - \frac{2\Gamma}{r^2}\right| + \left|\frac{\partial w}{\partial r}\right|\right)$$

K= 0.01

- Simulation parameters

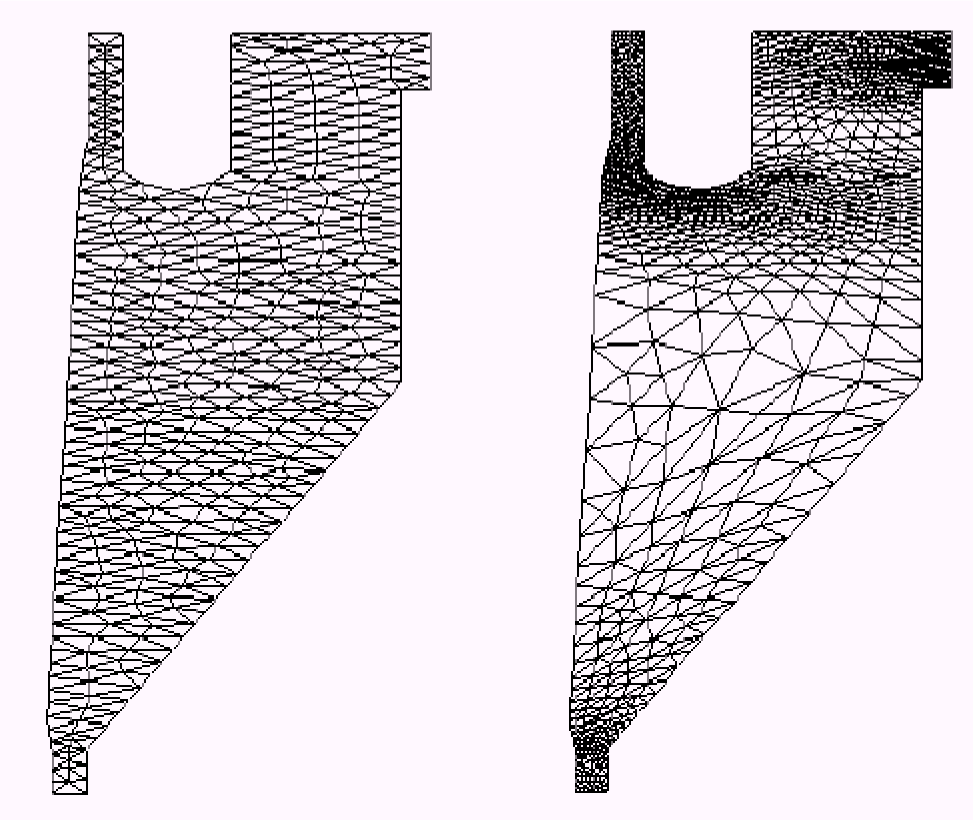
$$Re = 5.7 \times 10^5$$

$$\mathrm{dt} = 0.01 \ \text{seconds}$$

Results...

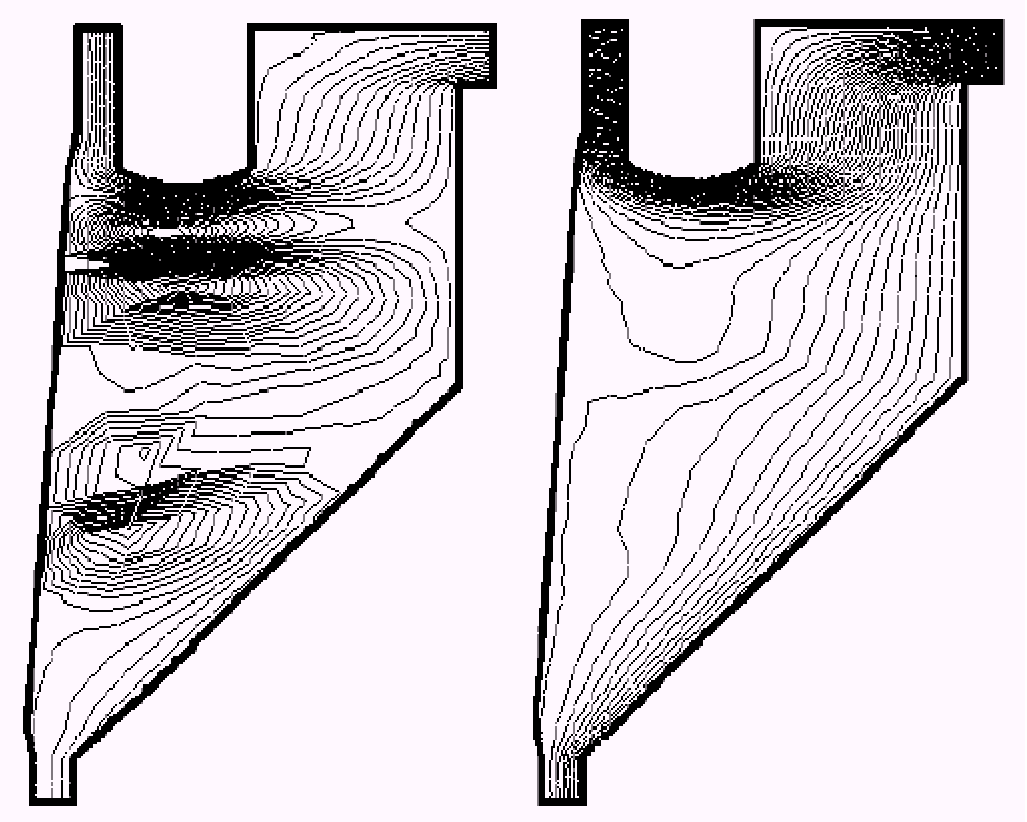
Dimension	cm.
Diameter hydrocyclone	50.80
Longitude cylindrical region	80.00
Feed diameter	13.13
Overflow diameter	11.43
Vortex diameter	26.67
Longitude vortex finder	31.80
Longitude conic region	116.80
Underflow diameter	6.35
Operation variable	m^3/s
Feed flux	0.02348
Underflow flux	0.00643

Results...



Adapted mesh: 1109 points, 1987 triangles

Results...



Stream-function solution with and without stabilization scheme.

Conclusions

- A stabilized FEM solution has been obtained using cilindrical coordinates on triangular adapted meshes
- The adjustement of the experimental form of a Rankine profile is obtained trough a turbulence model