Notes

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Abstract

These are the notes of lectures given by Professor M. Kurgansky in Atmospheric and Ocean Physics Department (DEFAO) on Concepción University, about some topics in hydrodynamics. Of course, these notes contain my own interpretation of his lectures.

1 Introduction

We consider the Lyapunov method by Arnold (1965), for study of the stability of steady flows [1] .

2 The dynamical setting

In the following analysis, we shall consider the following restrictions on our problems:

- no viscosity;
- all forces can be derived from a potential function $\mathbf{F} = -\nabla\phi$;
- flow is adiabatic.

We shall be interested first, in stationary flows, were the boundary condition for the velocity takes the form $\mathbf{v} \cdot \mathbf{n} = 0$ (with $\mathbf{n}$ being the normal vector to the boundary of the domain). Examples of these flows are the atmosphere

[1] The question whether a flow is stable or not, is not trivial. For example, consider the stationary problem that represents the dynamics of a rotating planet that is covered with a water layer. If the form of the geoid is not specified accurately, it is possible to obtain some spurious lateral forces.

(compressible fluid) and the ocean (nearly incompressible medium, that is $\nabla \cdot \mathbf{u} \approx 0$).

The equations of motion are used in the Lamb, or Gromeka-Lamb form, for an incompressible but heterogeneous by density fluids, that is $\mathbf{v} \cdot \nabla \rho = 0$

$$\begin{aligned} & \nabla \left(\frac{\mathbf{u}^2}{2}\right) + \mathbf{w}_a \times \mathbf{v} = -\frac{\nabla p}{\rho} - \nabla \phi \\ & \nabla \cdot \mathbf{v} = 0 \\ & \mathbf{v} \cdot \nabla \rho = 0 \end{aligned} \tag{1}$$

where $\mathbf{w}_a = 2\mathbf{\Omega} + \nabla \times \mathbf{v}$ is the absolute vorticity given in terms of the angular velocity $\mathbf{\Omega}$ and the local vorticity field $\mathbf{w} = \nabla \times \mathbf{v}$. In the above equations, we have reduced the continuity equation $\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0$ to $\nabla \cdot \mathbf{v} = 0$. The last equation comes from the adiabatic restriction for the energy equation.

Ertel (1942) and Truesdell (1951), following Ertel, derived the strict formalization of a discovery of Rossby, about conservation of potential vorticity $\Pi_\rho \equiv \frac{\mathbf{w}_a \cdot \nabla \rho}{\rho}$

$$\frac{D\Pi_\rho}{Dt} = 0 \tag{2}$$

Furthermore, for a stationary flow $\mathbf{v} \cdot \nabla \Pi_\rho = 0$. For an ideal fluid, we have that $\nabla B = \mathbf{v} \times \mathbf{w}_a$ (Batchelor, [1]), where $B \equiv \frac{\mathbf{v}^2}{2} + \frac{p}{\rho} + \phi$ is the *Bernoulli function*, as can be proved easily multiplying equation (1) scalarly by $\mathbf{v}$ and using $\mathbf{v} \cdot \nabla \rho = 0$. Then, the velocity vector $\mathbf{v}$ is orthogonal to the gradients of ρ, Π_ρ and B simultaneously. As these gradients are necessary coplanar, the streamline is given by intersection of any two of these three planes $\rho = const$, $\Pi_\rho = const$ and $B = const$. Therefor, if degeneration exist, we can express each of the invariant in terms of the other two. Consider, for example, that $B = B(\rho, \Pi_\rho)$. Then

$$\nabla B = \frac{\partial B}{\partial \rho} \nabla \rho + \frac{\partial B}{\partial \Pi_\rho} \nabla \Pi_\rho = \mathbf{v} \times \mathbf{w}_a - \frac{p}{\rho^2} \nabla \rho \tag{3}$$

where we have used that $\nabla \left(\frac{p}{\rho}\right) = \frac{\nabla p}{\rho} - \frac{p}{\rho^2} \nabla \rho$, and the equation (1). Multiplying the above equation by $\nabla \rho \times$, that is, taking the vector product $\nabla \rho \times \nabla B$, we obtain, using the vectorial identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$, the *form 1* of the equation of motion

$$\rho \mathbf{v} = \frac{1}{\Pi_\rho} \frac{\partial B}{\partial \Pi_\rho} \left(\nabla \rho \times \nabla \Pi_\rho\right) \tag{4}$$

The geometrical interpretation of this equation is that the flux $\rho\mathbf{v}$, formed by intersects of surfaces $\rho = const$ and $\Pi_\rho = const$, is inversely proportional to the transversal area of tubes or ρ and Π_ρ.

Other interesting relation can be obtained if we take a vector product $\nabla\Pi_\rho \times \nabla B$. In this case we obtain the *form II* of the equation of motion

$$\rho\mathbf{v} = -\left(\frac{\mathbf{w}_a \cdot \nabla\Pi_\rho}{\rho}\right)^{-1}\left(\frac{\partial B}{\partial \rho} + \frac{p}{\rho^2}\right)(\nabla\rho \times \nabla\Pi_\rho) \tag{5}$$

Comparing equations (4) and (5), we obtain a general expression for a kind of generalized potential vorticity (or secondary potential vorticity), that is expressed now in terms of the gradient of Π_ρ in place of the gradient of ρ

$$\frac{\mathbf{w}_a \cdot \nabla\Pi_\rho}{\rho} = -\left(\frac{1}{\Pi_\rho}\frac{\partial B}{\partial \Pi_\rho}\right)^{-1}\left(\frac{\partial B}{\partial \rho} + \frac{p}{\rho^2}\right) \tag{6}$$

In some cases, the above formulation can be treated in a more elegant fashion using the generalized Hamiltonian dynamics discovered by Nambu (1973). This formulation describes the evolution of a function $F(x, y, z, t)$, in terms of two *Hamiltonians* $H(x, y, z, t)$ and $G(x, y, z, t)$ in the form

$$\dot{F} = [F, H, G] \tag{7}$$

where $\dot{F} = \frac{dF}{dt}$ is a material time derivate of F and $[F, H, G] = \nabla F \cdot (\nabla H \times \nabla G) = \frac{\partial(F,H,G)}{\partial(x,y,z)}$ is a generalized Poisson bracket. In particular, if $F = F(x, y, z)$, where $\mathbf{r} = (x, y, z)$ is the position vector, we have $\nabla\mathbf{r} = \mathbf{I}$ and

$$\begin{aligned} \dot{x} &= \{H, G\}_{y,z} \\ \dot{y} &= \{H, G\}_{z,x} \\ \dot{z} &= \{H, G\}_{x,y} \end{aligned} \tag{8}$$

where $\{H, G\}_{a,b} = \frac{\partial H}{\partial a}\frac{\partial G}{\partial b} - \frac{\partial H}{\partial b}\frac{\partial G}{\partial a} \equiv \frac{\partial(H,G)}{\partial(a,b)}$ is the usual Poisson bracket. These three equations could be combined on a vector equation

$$\frac{d\mathbf{r}}{dt} = \nabla H \times \nabla G \tag{9}$$

Using $\mathbf{v} = \frac{d\mathbf{x}}{dt}$, from (9) we have that the fluid is incompressible, because $\nabla \cdot \mathbf{v} = 0$. For other side, from (7), of follow that H and G are constants of motion and $\mathbf{v} \cdot \nabla H = (\nabla H \times \nabla G) \cdot \nabla H = 0$ and similarly $\mathbf{v} \cdot \nabla G = 0$.

In general, the most general form of velocity field which satisfies $\nabla \cdot \mathbf{v} = 0$, $\mathbf{v} \cdot \nabla\rho = 0$ and $\mathbf{v} \cdot \nabla\Pi_\rho$ is given by

$$\mathbf{v} = \kappa(\rho, \Pi_\rho)(\nabla\rho \times \nabla\Pi_\rho)$$

This equation can be written in a Nambu form only when [2].

$$\kappa(\rho, \Pi_\rho) = \kappa_1(\rho)\kappa_2(\Pi_\rho) \tag{10}$$

with κ_1 and κ_2 being arbitrary functions. Now, it can be verified that $\dot{B} = \nabla B \cdot (\nabla\rho \times \nabla\Pi_\rho) = 0$ and $\dot{B} = \dot{\Pi}_\rho = 0$. In the rest of the chapter, we will not use the restriction (10), and will consider the more general case where the velocity field can be written as $\mathbf{v} = \kappa(\rho, \Pi_\rho)(\nabla\rho \times \nabla\Pi_\rho)$.

3 The stability of the system

We are interested in to investigate about the stability of the systems described above. First, we will need some preliminary lemmas.

Lemma *The mechanical energy, defined by*

$$E = \int \left(\frac{1}{2}\rho\mathbf{v}^2 + \rho\phi\right) dV \tag{11}$$

is a constat of motion

[2] In our original formulation of an ideal flow in terms of the invariants B, ρ and Π_ρ, we would need a Nambu formulation in four dimensions. The problem is however degenerate, in the sense that for the adiabatic case, we can consider only two independent invariants, (for example ρ and Π_ρ), because the third invariant can be written in terms of them.

Proof: Using the transport theorem[3] we have that $\frac{dE}{dt} = \int \rho \frac{d}{dt}\left(\frac{1}{2}\mathbf{v}^2 + \phi\right) dV = \int \rho \left(\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} + \frac{d\phi}{dt}\right) dV$. In the stationary case, $\rho\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = \rho\mathbf{v} \cdot \mathbf{v} \cdot (\nabla \cdot \mathbf{v}) = -\mathbf{v} \cdot \nabla p$. Then, $\frac{dE}{dt} = -\int \mathbf{v} \cdot (\nabla p) dV = \int \nabla \cdot (\mathbf{v}p) dV = 0$ if we choose the boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$ over the boundary.□

Lemma Consider a functional

$$F = \int \Phi(\rho, \Pi_\rho)\rho dV \tag{12}$$

Then, F is a constant of motion.

Proof: Applying the transport theorem, we have that

$$\frac{dF}{dt} = \int \left(\frac{\partial \Phi}{\partial \rho}\frac{d\rho}{dt} + \frac{\partial \Phi}{\partial \Pi_\rho}\frac{d\Pi_\rho}{dt}\right) \rho dV = 0$$

because ρ and Π_ρ are material constants themselves.□

We now apply the Arnold method, wish say that a necessary condition for stability, is that the functional $H = E + F$ calculated for a ensemble is an extremum. In more formal word, we have the following Lemma[4]

[3] The the Reynolds or transport theorem is
Theorem *If $f(\mathbf{x}, t)$ is a continuum function, then*

$$\frac{d}{dt} \int f \rho dV = \int \frac{df}{dt} \rho dV$$

where $\frac{d(\cdot)}{dt} = \frac{\partial(\cdot)}{\partial t} + \mathbf{v} \cdot \nabla(\cdot)$ is the material derivate of f.
Proof In general, it is not direct to apply directly the derivate on f, because the volume of integration dV can be changing in time. However, we can do the calculation in another configuration, where the volume dV_0 is constant. If J is the jacobian of this transformation, then $dV = JdV_0$ and

$$\frac{d}{dt} \int_V f \rho dV = \int_{V_0} \frac{d(f\rho J)}{dt} dV_0$$

As is demonstrate in Media Continuous Mechanics, $\frac{dJ}{dt} = J\nabla \cdot \mathbf{v}$, and the above equation can be rewritten as $\int_V \left(\rho\frac{df}{dt} + f\frac{d\rho}{dt} + f\rho\nabla \cdot \mathbf{v}\right) dV$. Finally, using the continuity equation $\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho\mathbf{v})$, we complete the proof.
[4] Assume that we are interested in to learn what happens with a flow in a particular point of space, say $\mathbf{x}$. If the fluid has a constant mechanical energy E, the velocity of the fluid in $\mathbf{x}$ will be $\mathbf{v}$, but if we change the total mechanical energy to $E + \Delta E$, we will obtain another velocity, say $\mathbf{v} + \Delta\mathbf{v}$ in $\mathbf{x}$. The same will occurs, if we change the density to $\rho + \Delta\rho$ or the pressure to $p + \Delta p$. The question then is, if we consider an collection or ensemble of systems, each of them having little variation in density, pressure and velocity respect to a given reference configuration, what is the total energy that will be the most probable? and also the another question: if a fluid

Lemma *If ρ and Π_ρ are two material constants of motion, and $E = \int \left(\frac{1}{2}\mathbf{v}^2 + \phi\right)\rho dV$ and $F = \int \Phi(\rho, \Pi_\rho)\rho dV$ are two functionals that represent the energy and a Casimir (that represent some symmetriy property of the system), then, the demand of vanishing of the first variation δH of the functional $H = E + F$, respective to arbitrary variations in the fluid dynamics fields $\mathbf{u} = \mathbf{u}_0 + \delta\mathbf{u}$, $\rho = \rho_0 + \delta\rho$ and $p = p_0 + \delta p$, is equivalent to the two following equations*

$$\rho\mathbf{v} = \Phi''_{\Pi\Pi}(\nabla\rho \times \nabla\Pi_\rho) \tag{13}$$

$$\frac{\mathbf{v}^2}{2} + \phi + \frac{\partial(\rho\Phi)}{\partial\rho} - \frac{\mathbf{w}_a \cdot \nabla\rho}{\rho}\rho\Phi''_{\Pi} - \frac{\mathbf{w}_a \cdot \nabla\Pi_\rho}{\rho}\rho\Phi''_{\Pi\Pi} + \Phi'_{\Pi}\Pi_\rho = 0 \tag{14}$$

where zero field variations at the borders have been assumed.

Proof: We consider a stationary flow $\mathbf{v}(\mathbf{x})$, $\rho(\mathbf{x})$ and consider the variations $\mathbf{\Pi}_\rho(\mathbf{x})$ as independent. Then, the first variation of $H(\mathbf{v}, \rho, \mathbf{\Pi}_\rho)$ is

$$\delta H = \int \left(\rho\mathbf{v} \cdot \delta\mathbf{v} + \left(\frac{\mathbf{v}^2}{2} + \Phi\right)\delta\rho + \rho\Phi'_\rho\delta\rho + \rho\Phi'_\Pi\delta\Pi_\rho + \Phi\delta\rho\right) dV = 0 \tag{15}$$

where we have used the notation $\Phi'_\rho \equiv \frac{\partial\Phi}{\partial\rho}$ and $\Phi'_\Pi \equiv \frac{\partial\Phi}{\partial\Pi_\rho}$. For other side, we have that

$$\delta\Pi_\rho = \underbrace{\frac{\mathbf{w}_a \cdot \nabla\delta\rho}{\rho}}_{I} + \underbrace{\frac{(\nabla \times \delta\mathbf{v}) \cdot \nabla\rho}{\rho}}_{II} + \underbrace{-\frac{\mathbf{w}_a \cdot \nabla\rho}{\rho^2}}_{III} \tag{16}$$

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have some energy E, how is the stability of the system for arbitraries changes in the properties of the fluid? The above questions, can be analyzed with a variational formalism. The idea is search an extremum value for a functional $H = E + F$ where E is the mechanical energy and F are other conserved quantities.
Remark 1 The new functions now can be considered fields, and for this reason we will need to do some mathematical operations (as differentiation by example), using the language of a field theory.
Remark 2 In some circumstances it might be necessary consider a larger number of restrictions over each one of the components of the ensemble, as for example in the case when each one the systems have associated others conserved quantities that the mechanical energy. This conserved quantities can be expressions of some internal symmetries.
The problem can be then posed formally as a variational problem, where we are trying to find the more probable energy, but with some constrains that can be introduced in the analysis through the some Lagrange multipliers. This technique, developed by Lagrange, when is applied to the fluid problem, is denominated the Arnold method, because he was the first (around 1965) in to apply this in the study of hydrodynamical stability.

Substituting (16) in (15), we have that the term $\frac{\mathbf{w}_a \cdot \nabla \rho}{\rho^2} \delta\rho$ can be written as $-\Pi_\rho \frac{\delta\rho}{\rho}$, because $\mathbf{w}_a = 2\mathbf{\Omega} + \nabla \times \mathbf{v}$. The term in (20) proportional to (I) is, when integrated by parts, equal to

$$\int \rho \Phi'_\Pi \frac{\mathbf{w}_a \cdot \nabla \delta\rho}{\rho} dV = \int \nabla \cdot \left(\Phi'_\Pi \mathbf{w}_a \delta\rho \right) dV - \int (\mathbf{w}_a \cdot \nabla\rho) \Phi''_\Pi \delta\rho dV \\ - \int (\mathbf{w}_a \cdot \nabla \Pi_\rho) \Phi''_{\Pi\Pi} \delta\rho dV$$

In the last equation, the first term is nil because by Gauss divergence theorem, this term is equal to $\int \Phi'_\Pi \delta\rho \mathbf{w}_a \cdot \mathbf{n} d\sigma$ and $\delta\rho$ that is nil in the border. For the term proportional to (II), we use the vectorial identity $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b}$. Then

$$\int \Phi'_\Pi (\nabla \times \delta\mathbf{v}) \cdot \nabla\rho dV = -\int \nabla \cdot (\Phi'_\Pi \nabla\rho \times \delta\mathbf{v}) dV + \int \Phi''_{\Pi\Pi} \nabla \Pi_\rho \cdot (\nabla\rho \times \delta\mathbf{v}) dV \\ + \int \Phi''_\Pi \nabla\rho \cdot (\nabla\rho \times \delta\mathbf{v}) dV$$

Again, on the right side, the first term is nil due to boundary conditions. The third right hand term is also nil due to $\nabla\rho \cdot \nabla\rho \times \delta\mathbf{v} \equiv 0$. Then, the first variation of δH can be written as

$$\delta H = \int \left(\rho\mathbf{v} + \Phi''_{\Pi\Pi} (\nabla \Pi_\rho \times \nabla\rho) \right) \cdot \delta\mathbf{v} dV \\ \int \left(\frac{\mathbf{v}^2}{2} + \phi + \frac{\partial(\rho\Phi)}{\partial\rho} - \frac{\mathbf{w}_a \cdot \nabla\rho}{\rho} \rho \Phi''_\Pi - \frac{\mathbf{w}_a \cdot \nabla\Pi_\rho}{\rho} \rho \Phi''_{\Pi\Pi} + \Phi'_\Pi \Pi_\rho \right) \delta\rho dV$$

As $\delta\rho$ and $\delta\mathbf{v}$ are independent, we have that

$$\rho\mathbf{v} = \Phi''_{\Pi\Pi} (\nabla\rho \times \nabla\Pi_\rho) \tag{17}$$

$$\frac{\mathbf{v}^2}{2} + \phi + \frac{\partial(\rho\Phi)}{\partial\rho} - \frac{\mathbf{w}_a \cdot \nabla\rho}{\rho} \rho \Phi''_\Pi - \frac{\mathbf{w}_a \cdot \nabla\Pi_\rho}{\rho} \rho \Phi''_{\Pi\Pi} + \Phi'_\Pi \Pi_\rho = 0 \tag{18}$$

This completes the proof.□

The next task is to find an expression for the functional Φ, which is valid for fluids as we are studding. As Φ is a additional constrains to the dynamical systems, it shluld need be constructed in terms of invariants of the flow.

Lemma *For a ideal and heterogenous flow, a functional* Φ *can be constructed in terms of Bernoulli function* B *in the form*

$$\Phi = \Pi_\rho \int \frac{B - B_0}{\Pi_\rho^2} d\Pi_\rho \tag{19}$$

Conversely, we can express the Bernoulli function in terms of Φ as

$$B = \Phi'_{\Pi}\Pi_{\rho} - \Phi + \Psi(\Pi_{\rho}) \tag{20}$$

where Ψ is an arbitrary function.

Proof: This kind of fluid, satisfies the equations (5) and (6). Comparing (5) with (17) we have that

$$\Phi''_{\Pi\Pi} = \frac{1}{\Pi} B_{\Pi} \tag{21}$$

The second condition (14) impose a constrain over B

$$(\rho B)_{\rho} = (\rho \Phi_{\Pi}{}' \Pi_{\rho} - \rho\Phi)_{\rho} \tag{22}$$

Integrating this last equation, we can write B in terms of Φ in the form

$$B = \Phi'_{\Pi}\Pi_{\rho} - \Phi + \Psi(\Pi_{\rho}) \tag{23}$$

where Ψ is an arbitrary function. Now, from (6) and (21) we have that

$$\frac{\mathbf{w}_a \cdot \nabla \Pi_{\rho}}{\rho} = -(\Phi''_{\Pi\Pi})^{-1} \left(B_{\rho} + \frac{p}{\rho^2} \right) \tag{24}$$

If replace (24) in (23) for conclude that $\Psi'_{\rho} = 0$, that is Ψ is a constant. Renaming Φ by B_0 in (23) and do an integration, we obtain finally (19).□

References

[1] Batchelor G.K., (1967), *An introduction to fluid mechanics*, Cambridge University Press.

[2] Marsden J., (1997), *Lectures on Mechanics*, second edition.

[3] Nambu Y., (1973), *Generalized Hamiltonian Dynamics*, Phys. Rev. D, Vol 7, number 8, pp 2405-2412.