INTRODUCTION TO TROPICAL GEOMETRY

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INTRODUCTION

The aim of these notes is to introduce the subject of tropical geometry and subsequently describe the correspondence between algebraic varieties and tropical varieties.

The tropical algebra is introduced in section 1. This allows one to define polynomials and to prove a fundamental theorem of algebra in the tropical context. Tropical lines are the argument of section 2.

1. TROPICAL ALGEBRA

Let us consider the set of real numbers $\mathbb{R}$ endowed with two operations:

$$x \odot y = x + y, \qquad x \oplus y = \min\{x, y\}$$

for any $x, y \in \mathbb{R}$. The new operations are commutative, associative and $\odot$ is distributive with respect to $\oplus$:

$$\begin{array}{ll} x \odot y = y \odot x & x \oplus y = y \oplus x, \\ x \odot (y \odot z) = (x \odot y) \odot z & x \oplus (y \oplus z) = (x \oplus y) \oplus z, \\ x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z). & \end{array}$$

The $\odot$ unity of the semiring is 0, while there is no $\oplus$ unity. For this reason we will consider the extension

$$\mathbb{R}_{\min} = \mathbb{R} \cup \{+\infty\},$$

and for any $x \in \mathbb{R}_{\min}$ we have $x \oplus +\infty = x$.

1.1. *Definition.* — The triple $(\mathbb{R}_{\min}, \odot, \oplus)$ is called the *tropical semiring*.

Observe that $x \oplus x = x$ for any $x \in \mathbb{R}_{\min}$, i.e. the sum is *idempotent*. We will write $x^n = x \odot \ldots \odot x$ for the n-times product of x with itself.

1.2. An univariate polynomial is an object of the form:

$$P(x) = a_n \odot x^n \oplus \cdots \oplus a_1 \odot x \oplus a_0.$$

Of course as soon as we have polynomials, it is possible to ask for solutions of algebraic equations. Consider for example the equation

$$a \odot x \oplus b = 0.$$

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Observe that there are many differences with respect to the classical case. One is that the right hand side of the equation is no longer the additive unity. Another is that there is no additive inverse of b. The preceding equation is equivalent to

$$\min\{a + x, b\} = 0,$$

so that the solution is given by:

$$\begin{cases} x = -a & \text{if} \quad b > 0 \\ x \geq -a & \text{if} \quad b = 0 \\ \quad \emptyset & \text{if} \quad b < 0 \end{cases}$$

The fact that there are infinitely many or no solutions for an univariate polynomial suggests the idea that probably we are not looking at the right definition of equation.

Looking at the general case, let

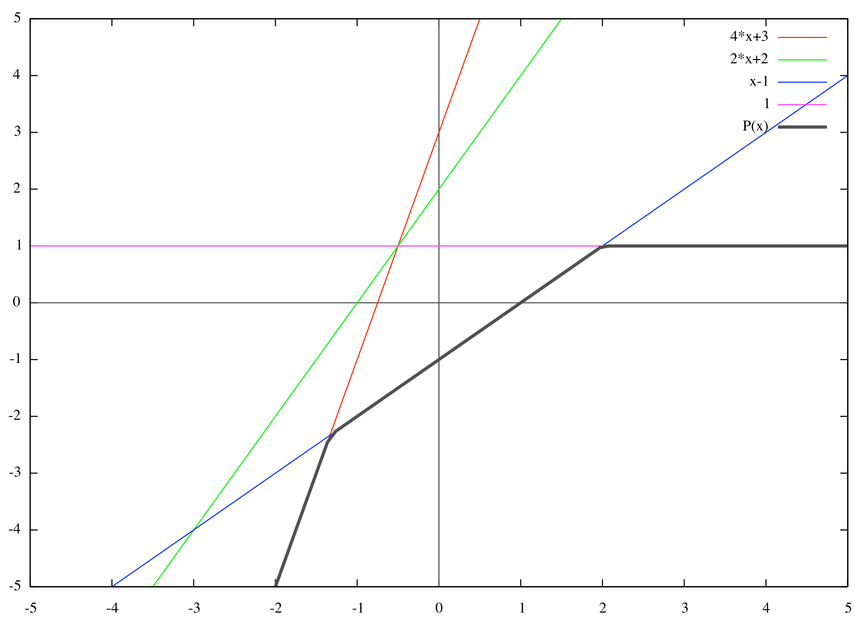
$$P(x) = \bigoplus_{i=0}^{n} a_i \odot x^i,$$

then the equation $P(x) = 0$ is equivalent to

$$\min\{a_n + nx, a_{n-1} + (n-1)x, \ldots, a_1 + x, a_0\} = 0.$$

This last equation can be solved graphically by considering the set of lines $y = a_i + ix$ for $i = 0, \ldots, n$ and looking at their minimum values.

1.3. *Example.* — Consider the polynomial $P(x) = 3 \odot x^4 \oplus 2 \odot x^2 \oplus -1 \odot x \oplus 1$ and look at the corresponding lines:



In this case the "solution" of the equation seems to be given by $x = 1$. Observe that the term $2 \odot x^2$, which corresponds to the line $2 + 2x$ is always bigger than the corresponding minimum. This means that the values attained by the two functions

$$P(x) = 3 \odot x^4 \oplus 2 \odot x^2 \oplus -1 \odot x \oplus 1, \qquad Q(x) = 3 \odot x^4 \oplus -1 \odot x \oplus 1$$

are the same for any value of x. In particular two different polynomials (like P and Q) can be identical as functions.

The graph of $y = P(x)$ is a piecewise linear function with two vertices at $x = -4/3$ and $x = 2$. What we want to prove here is that

$$P(x) = 3 \odot (x \oplus -4/3)^3 \odot (x \oplus 2).$$

First of all observe that

$$(x \oplus a)^n = x^n \oplus a^n,$$

this can be proved by induction on n. The case $n = 1$ is obviously true, moreover

$$\begin{aligned}
(x \oplus a)^n &= (x \oplus a) \odot (x \oplus a)^{n-1} = (x \oplus a) \odot (x^{n-1} \oplus a^{n-1}) \\
&= x^n \oplus a \odot x^{n-1} \oplus x \odot a^{n-1} \oplus a^n \\
&= x^n \oplus a^n
\end{aligned}$$

where the last equality comes from the fact that

$$\min\{nx, a + (n-1)x, x + (n-1)a, na\} = \min\{nx, na\}.$$

The analysis carried out for $P(x)$ suggests to define the *roots* of a polynomial as the set of vertices appearing in the graph of the polynomial itself. It remains to explain where does the exponent of the first linear form comes from. Let $p := (-4/3, -7/3)$, $q := (2, 1)$ and let

$$M_p := \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \qquad M_q := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

be the matrices whose columns are the directions of the two lines coming out from p and q respectively. Observe that the determinants of these two matrices are exactly the exponents of the two linear forms used in the decomposition of $P(x)$ as a product of linear terms.

1.4. *Proposition.* — Any tropical polynomial of degree n factorizes as the product of n linear polynomials.

Proof. Let $P(x) = \bigoplus_{i=0}^{n} a_i \odot x^i$ and observe that its graph is generated by half lines and segments. Let $y = a_j + jx$ be the consecutive line to $y = a_i + ix$ in the graph of $y = P(x)$, then the matrix of the directions at the corresponding root p_{ij} is

$$M_{p_{ij}} = \begin{pmatrix} 1 & 1 \\ i & j \end{pmatrix}$$

and its determinant is $j - i$. To any such p_{ij} we associate the power $(x \oplus \frac{a_i - a_j}{j-i})^{j-i}$. Now let $i_1 < \cdots < i_r$ be the indices of the roots of $P(x)$. It is a straightforward calculation (done comparing with the graph of $P(x)$) to see that

$$P(x) = a_n \odot \left(x \oplus \frac{a_{i_r} - a_{i_{r-1}}}{i_r - i_{r-1}} \right)^{i_r - i_{r-1}} \odot \cdots \odot \left(x \oplus \frac{a_{i_2} - a_{i_1}}{i_2 - i_1} \right)^{i_2 - i_1}.$$

$\square$

1.5. **Exercises.**

(1) Verify the associativity, commutativity and distributive properties of $(\odot, \oplus)$.

(2) Decide if the right hand sides of the following identities can be simplified or not.
$(x \oplus y) \odot (x \oplus y) = x^2 \oplus x \odot y \oplus x \odot y \oplus y^2$
$(x \oplus y) \odot (x \oplus -y) = x^2 \oplus x \odot y \oplus x \odot -y \oplus 0$

(3) Prove the following properties of the exponential function:

$$\exp(x \oplus y) = \exp(x) \oplus \exp(y) \qquad \exp(x \odot y) = \exp(x)\exp(y).$$

(4) Find all the continuous functions $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

$$\exp(f(x, y)) = f(\exp(x), \exp(y)).$$

(5) Prove that

$$3 \odot x^4 \oplus 2 \odot x^2 \oplus -1 \odot x \oplus 1 = 3 \odot x^4 \oplus -1 \odot x \oplus 1$$

without making use of the graph.

(6) Prove that $3 \odot x^4 \oplus 2 \odot x^2 \oplus -1 \odot x \oplus 1 = 3 \odot (x \oplus -4/3)^3 \odot (x \oplus 2)$.

(7) Factorize the expression $x^n \oplus x^{n-1} \oplus \cdots \oplus x \oplus 1$.

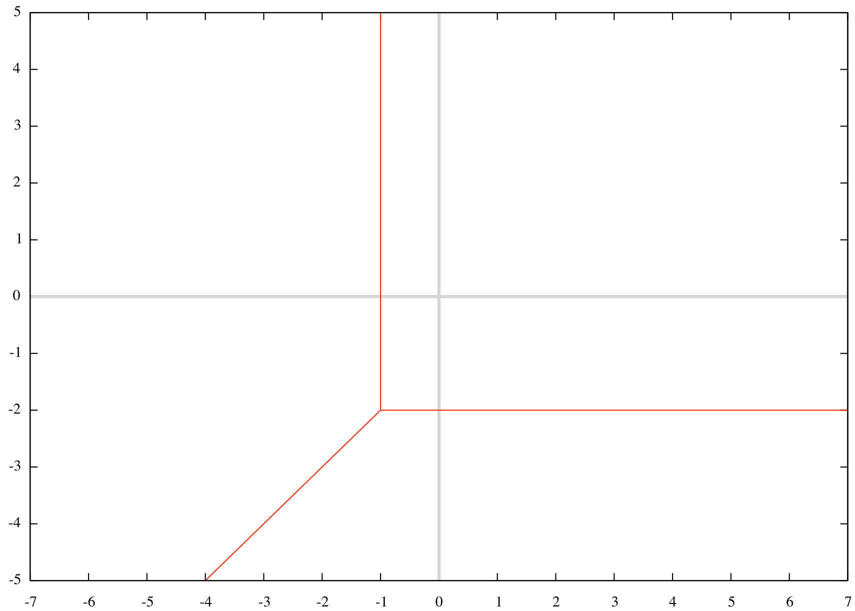
2. Tropical lines in $\mathbb{R}^2$

We explore the idea of defining the "zero locus" of a tropical polynomial to be the set of points where the minimum of its linear forms is attained twice. This allows us to define tropical lines and study their main properties.

2.1. *Definition.* — A *tropical line* $a \odot x \oplus b \odot y \oplus c$ is the set of points where the minimum of the linear forms

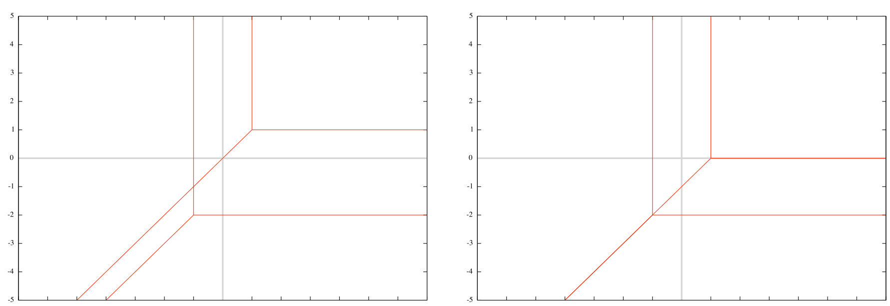
$$\min\{a + x, b + y, c\}$$

is attained twice. For example consider the tropical line given $2 \odot x \oplus 3 \odot y \oplus 1$, its graph is given below.



2.2. *Exercise.* — Prove that the graph of any tropical line $a \odot x \oplus b \odot y \oplus c$, with $a \odot b \odot c \neq +\infty$, is the union of three half lines departing from a common point with directions: $(1,0), (0,1), (-1,-1)$.

As for the intersection of two tropical lines we can distinguish two situations which are shown in the picture:



The first picture shows something already expected, i.e. two tropical lines which intersect at one point. This is no longer the case for the second picture. The equations of the lines contained in this picture are:

$$\begin{cases} 1 \odot x \oplus 2 \odot y \oplus 0 \\ 0 \odot x \oplus 1 \odot y \oplus 1 \end{cases}$$

Observe that the second column is a multiple of the first:

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 1 \odot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

This means that the two tropical lines are the equivalent of two classical parallel lines. The tropical matrix

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

has rank one. It is natural to ask if there is a way of detecting this by using determinants. The solution is offered by the following.

2.3. *Definition.* — The *tropical determinant* of a 2×2 matrix is defined as

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \odot a_{22} \oplus a_{12} \odot a_{21}.$$

Moreover, we say the matrix is T-singular if the minimum is attained twice, i.e. $a_{11} + a_{22} = a_{12} + a_{21}$.

The matrix of the preceding example is T-singular. This is a general phenomenon which can can be summarized by the following:

2.4. *Proposition.* — Two tropical lines $a_{11} \odot x \oplus a_{12} \odot y \oplus a_{13}$ and $a_{21} \odot x \oplus a_{22} \odot y \oplus a_{23}$ intersect in one point if and only if all the minors of the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

are non T-singular.

2.5. **Exercises.**

(1) Prove Proposition 2.4.

(2) Determine the family of tropical lines through the point $(1, 2)$.

(3) Prove that through any two general distinct points there is one and only one tropical line.

(4) Compute the tropical product

$$\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix}$$

(5) Evaluate the tropical determinant of the following matrices and decide which ones are singular.

$$\begin{pmatrix} 2 & 3 & 5 \\ 7 & 11 & 13 \\ 17 & 19 & 23 \end{pmatrix} \quad \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

References

[1] Richter-Gebert, J. Sturmfels, B. and Theobald, T. *First steps in tropical geometry.* math.AG/0306366.

[2] Speyer, D. and Sturmfels, B. *Tropical Mathematics.* math.CO/0408099

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