Curves Seminar - Fall 2005

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Lecture 3

1. Two equivalent conjectures

We start this section by recalling the classical Gimigliano-Harbourne-Hirschowitz conjecture:

Conjecture. *A non-empty linear system $\mathcal{L}$ is special if and only if there exists a (-1)-curve E such that $\mathcal{L}E \leq -2$.*

We recall that a linear system $\mathcal{L}_2(d; m_1, \ldots, m_r)$ is in standard form if $m_1 \geq \ldots \geq m_r \geq 0$ and $d \geq m_1 + m_2 + m_3$.

Conjecture 1.1. *A linear system $\mathcal{L}$ in standard form is non-special.*

It is easy to see that if Conjecture 1.1 is true than there exists an algorithm for deciding if a linear system is special or not. Moreover one has the following.

Theorem 1.2. *Conjecture 1.1 is equivalent to the G.H.H. conjecture.*

Proof. Suppose that $\mathcal{L}$ is in standard form then, by Proposition 3.4 of Lecture 2, one has that $\mathcal{L}E \geq 0$ for any (-1)-curve E, hence by G.H.H. it is non-special.

Suppose now that $\mathcal{L}$ is a linear system such that $\mathcal{L}E \geq -1$ for each (-1)-curve E. Observe that if $\mathcal{L}E = -1$ then

$$\begin{aligned} v(\mathcal{L}) &= v(\mathcal{L} - E) + v(E) + (\mathcal{L} - E)E \\ &= v(\mathcal{L} - E), \end{aligned}$$

so that $\mathcal{L} - E$ is a new system with the same virtual and effective dimension of $\mathcal{L}$. After removing from $\mathcal{L}$ all the (-1)-curves E such that $\mathcal{L}E = -1$ we obtain a new system $\mathcal{L}'$ which has non-negative intersection with any (-1)-curve and has the same virtual and effective dimension of $\mathcal{L}$. If $\mathcal{L}'$ is not in standard form, then by applying a quadratic transformation σ based on the three points of biggest multiplicities we can decrease its degree. Observe that $\sigma^*(\mathcal{L}')$ can not have negative multiplicities, since this would imply that $\mathcal{L}'$ intersects negatively a line through two points (which is a (-1)-curve). Proceeding in this way, after a finite number of steps, $\mathcal{L}'$ transforms into a linear system $\mathcal{L}''$ which is in standard form. Since $v(\mathcal{L}) = v(\mathcal{L}'')$ then $\mathcal{L}$ is non-special. □

2. Some results on the structure of special linear systems

In this section we recall some evidences for the G.H.H. conjecture.

The first idea is due to Harbourne.

Theorem 2.1. *The G.H.H. conjecture is true for linear systems of the form* $\mathcal{L}_2(d; m_1, \dots, m_r)$ *with* $r \leq 9$.

Instead of giving an idea of the proof of this theorem, which would go beyond the possibilities of these lectures, we will show another interesting fact about the case $r = 8$.

Proposition 2.2. *The set of* (-1)*-curves of type* $\mathcal{L}_2(\delta; \mu_1, \dots, \mu_8)$ *is in correspondence with a finite root system of type* E_8.

The anticanonical class $-K = \mathcal{L}_2(3; 1^8)$ has self-intersection $K^2 = 1$. To each (-1)-curve E associate the class $v_E := K + E$, then $v_E K = 0$ and $v_E^2 = -2$. Since the intersection form has signature $(1, 8)$, then the form is negative-definite on the orthogonal of a vector of positive length. This implies that

$$\#\{v \in K^\perp \cap \mathbb{Z}^9 \mid v^2 = -2\} < \infty.$$

The preceding is a structure of root system and moreover by considering the following elements

$$E_0 - E_1 - E_2 - E_3, \text{ and } E_i - E_{i+1} \text{ for } 1 \leq i \leq 7$$

one can see that it corresponds to E_8. The important fact is that the set Ω is in one-to-one correspondence with that of (-1)-curves. In fact given $v \in \Omega$ the class $E := v - K$ satisfies $E^2 = EK = -1$. Such a class corresponds to an effective curve, since

$$v(E) = \frac{E^2 - EK}{2} = 0.$$

Moreover if $E = E_1 + E_2$ is sum of two curves, then $1 = (E_1 + E_2)(-K)$ which means that at least one of the two curves must satisfy $E_i(-K) = 0$ which is not possible.
This second result is due to F. Cioffi, C. Ciliberto, R. Miranda and F. Orecchia.

Theorem 2.3. *The G.H.H. conjecture is true for linear systems of the form* $\mathcal{L}_2(d; m^r)$ *with* $m \leq 20$.

This result is due to S. Yang

Theorem 2.4. *Suppose that the G.H.H. conjecture is true for linear systems of the form* $\mathcal{L}_2(d; m_1, \dots, m_r)$ *with* $m_i \leq M$ *and* $d \leq f(M)$*, then it is true for* $m_i \leq M$ *and all* d.

She gives an explicit expression for $f(M)$ and by means of this she is able to prove the following

Theorem 2.5. *The G.H.H. conjecture is true for linear systems of the form* $\mathcal{L}_2(d; m_1, \dots, m_r)$ *with* $m_i \leq 7$.