Curves Seminar - Fall 2005

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Lecture 1

1. Linear systems through multiple points in $\mathbb{P}^2$

In what follows the ground field is assumed to be $\mathbb{C}$.

Consider a set of points $p_1, \ldots, p_r \in \mathbb{P}^2$ and for each of these points fix a non negative integer m_i. The main problem we are interested in is the following:

> Determine the dimension of the vector space of polynomials of degree d which have multiplicity m_i at each p_i.

This is equivalent to evaluate

$$\dim_{\mathbb{C}} \langle f \in I_{p_1}^{m_1} \cap \ldots \cap I_{p_r}^{m_r} \mid \deg(f) = d \rangle$$

where $I_{p_i}^{m_i}$ is the m_i power of the maximal homogeneous ideal defining p_i.

We will denote with $\mathcal{L}_2(d; m_1, \ldots, m_r)$ the projective space of such polynomials. The reason for this notations is that we are mainly interested in plane curves and two polynomials f and af define the same algebraic curve provided that $a \in \mathbb{C}^*$.

Remark 1.1. In case two or more multiplicities will be equal we will denote by the symbol $m_i^{a_i}$ a set of a_i points of multiplicity m_i. In this way a conic through 5 simple points will be denoted by $\mathcal{L}_2(2; 1^5)$.

Let $p = [0 : 0 : 1]$ then, in the affine chart (x, y), one has $I_p^m = \langle x^m, x^{m-1}y, \ldots, y^m \rangle$. For any $f \in \mathbb{C}[x, y]$ write $f = \sum f_i$ where f_i is homogeneous of degree i, then $f \in I_p^m$ if and only if $f_0 = \ldots = f_{m-1} = 0$. These equations are equivalent to a set of $\binom{m+1}{2} = m(m+1)/2$ linear equations in the coefficients of f.

In this way, for any linear system $\mathcal{L}_2(d; m_1, \ldots, m_r)$ we can try to evaluate its dimension by means of the following formula:

$$v(\mathcal{L}) = \binom{d+2}{2} - \sum_{i=1}^{r} \binom{m_i+1}{2} - 1. \tag{1}$$

This formula gives the actual dimension of $\mathcal{L}$ only if the linear equations on the coefficients of the general f are independent. It turns out that in general this fact is not true (see the examples with 2 or 5 points in Geramita's Lecture 5) . For this reason $v(\mathcal{L})$ will be called the *virtual dimension* of $\mathcal{L}$.

From the preceding definition, it is evident that

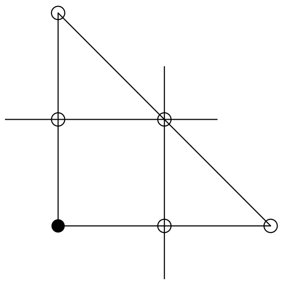
$$v(\mathcal{L}) \leq \dim \mathcal{L}.$$

If the inequality is strict and $\mathcal{L}$ is not empty then it is called a *special linear system.*

Example 1.2. Let $\mathcal{L}_2(2; 2^2)$ be the linear system of conics through two double points. By means of a linear change of coordinate we may assume that the two points are $[1 : 0 : 0]$ and $[0 : 1 : 0]$.

In the vector space of conics $\langle x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2\rangle$ those with a double point in $[1:0:0]$ are $\langle x_1^2, x_1x_2, x_2^2\rangle$, while those with a double point in $[0:1:0]$ are $\langle x_0^2, x_0x_2, x_2^2\rangle$. Therefore the space of conics with the two double points is $\langle x_2^2\rangle$, i.e. it is given by the "double line" through the two points. By means of formula (1) we obtain $v(\mathcal{L}) = -1$ even if $\mathcal{L}$ is not empty.

Exercise 1.3. Find a relationship between the preceding example and the following graph:



Then generalize the preceding example to the case of $\mathcal{L}_2(d; m_1, m_2, m_3)$.

Remark 1.4. Observe that if $d \geq \sum m_i - 1$ then $v(\mathcal{L}) = \dim \mathcal{L}$ as can be proved by putting all the points on a line.

In general $\dim \mathcal{L}_2(d; m_1, \ldots, m_r)$ depends on the position of the points. We define a set of r points to be in *general position with respect to* $\mathcal{L}$ if $\dim \mathcal{L}_2(d; m_1, \ldots, m_r)$ is maximal with respect to all the possible choices of r points in $\mathbb{P}^2$.

Example 1.5. Let p_1, p_2, p_3 be three points laying on a line, then they are in general position with respect to $\mathcal{L}_2(2; 1^3)$ but they are not in general position with respect to $\mathcal{L}_2(1; 1^3)$.

The preceding example suggests the following:

Definition 1.6. A set of points $p_1, \ldots, p_r$ is in *general position* if it in general position with respect to any $\mathcal{L}$.

The following facts are enunciated without a proof:
1. For any r there exist $p_1, \ldots, p_r \subset \mathbb{P}^2$ points in general position.
2. If $r \leq 8$ then a set of r points is in general position if no three of them lie on a line and no six of them lie on a conic.
3. For any m there are infinitely many distinct configurations of 9 points such that

$$\dim \mathcal{L}_2(3t; t^9) = \begin{cases} 0 & \text{if} \quad t < m \\ 1 & \text{if} \quad t = m \\ \geq 1 & \text{if} \quad t > m \end{cases}$$

even if $v(\mathcal{L}_2(3t; t^9)) = 0$ for any $t \in \mathbb{N}$.
It is not difficult to show that there are infinitely many special linear systems even if the points are in general position.

Example 1.7. Consider the linear system $\mathcal{L} := \mathcal{L}_2(2t; 2t-2, 2^{2t})$, its virtual dimension is given by $(2t+1)(2t+2)/2 - (2t-2)(2t-1)/2 - 6t - 1 = -1$. The linear system $\mathcal{M} := \mathcal{L}_2(t; t-1, 1^{2t})$ has virtual dimension $v(\mathcal{M}) = (t+1)(t+2)/2 - (t-1)t/2 - 2t - 1 = 0$ and so it is not empty. Observe that if $C \in \mathcal{M}$ then $2C \in \mathcal{L}$ and this means that $\mathcal{L}$ is not empty and thus it is special.

2. Numerical invariants of linear systems

Let D, D' be two plane algebraic curves and let $p \in D \cap D'$. It is possible to define an intersection index $i_p(D \cap D')$ in the following way:
If D and D' intersect transversely at p, then $i_p(D \cap D') = 1$. Moreover if D_λ ($\lambda \in \mathbb{C}$) is a family of plane curves such that $\deg D_\lambda = \deg D$ and $D_0 = D$, then for $|\lambda|, \epsilon \ll 1$

$$i_p(D \cap D') = \sum_{q \in \mathrm{B}(p,\epsilon)} i_q(D_\lambda \cap D')$$

where $\mathrm{B}(p, \epsilon)$ is the two dimensional complex ball of center p and radius ϵ. It is possible to prove that, by means of the preceding definition, the intersection index of two algebraic curves at a point is well defined and the following holds:

Theorem 2.1 (Bezout)**.** *Let D, D' be two algebraic plane curves of degree d and d' respectively without common components, then*

$$\sum_{p \in D \cap D'} i_p(D \cap D') = dd'$$

Let $\mathcal{L} := \mathcal{L}_2(d; m_1, \ldots, m_r)$ respectively $\mathcal{L}' := \mathcal{L}_2(d'; m'_1, \ldots, m'_r)$, then the preceding theorem is the key for defining the following intersection product between linear systems:

$$\mathcal{L}\mathcal{L}' = dd' - \sum_{i=1}^{r} m_i m'_i.$$

In fact it is easy to see that if D and D' are two curves with multiplicity m respectively m' at a point p, then $i_p(D \cap D') \geq mm'$. This implies that if $D \in \mathcal{L}$ and $D' \in \mathcal{L}'$, then the preceding product gives a measure of how many intersections D and D' have outside the multiple points $m_1, \ldots m_r$.

Proposition 2.2. *If $\mathcal{L}\mathcal{L}' < 0$ then the two systems have a common component.*

Proof. Let D and D' be defined as before, then

$$dd' = \sum_{p \in D \cap D'} i_p(D \cap D') \geq \sum_{i=1}^{r} i_{p_i}(D \cap D') = \sum_{i=1}^{r} m_i m'_i$$

unless the two curves have a common component. □

Corollary 2.3. *If the general element $D \in \mathcal{L}$ is irreducible and $\mathcal{L}\mathcal{L}' < 0$ then* $\dim \mathcal{L} = 0$ *and D is a fixed component of $\mathcal{L}'$.*

Observe that $2v(\mathcal{L}) - \mathcal{L}^2 = \mathcal{L}_2(3; 1^r)\mathcal{L}$ by formula 1 and this suggest to give a special name to the linear system $\mathcal{L}_2(3; 1^r)$. This system is classically denoted by $-K$ and it takes the name of "anticanonical system". By means of this definition, the formula for the virtual dimension is given by:

$$v(\mathcal{L}) = \frac{\mathcal{L}^2 - \mathcal{L}K}{2}. \tag{2}$$

This formula together with the intersection form are the basic tools needed to describe the main conjecture about the structure of special linear systems. But first of all one needs to introduce a particular class of linear systems.

3. (-1)-curves

The preceding formula for the virtual dimension of a linear system $\mathcal{L}$ allows one to construct some special linear systems.

Example 3.1. The system $\mathcal{L} = \mathcal{L}_2(2t; 2t-2, 2^{2t})$ of example 1.7 is not empty and special. We already observed that $\mathcal{L} \supseteq 2\mathcal{M}$ where $\mathcal{M} = \mathcal{L}_2(t; t-1, 1^{2t})$, here we want to prove that $\mathcal{L} = 2\mathcal{M}$. First of all observe that since the general curve of $\mathcal{L}_2(t; t-1)$ is irreducible and reduced and the points are in general position then $\mathcal{M}$ is irreducible and reduced.
The self intersection $\mathcal{M}^2 = -1$ and Corollary 2.3 imply that $\dim \mathcal{M} = 0$. Let $E \in \mathcal{M}$. From the irreducibility of E and the fact that $E\mathcal{L} = -2$ we deduce that E is a fixed component of $\mathcal{L}$, but removing this component from $\mathcal{L}$ one obtains $\mathcal{M}$ as a residual system and this means that $\mathcal{L} = 2\mathcal{M}$.

The key point in the preceding example is that $E \in \mathcal{M}$ is irreducible and reduced with $E^2 = -1$ and $v(E) = 0$. In this case, the linear system containing the curve $2E$ and denoted by $|2E|$, is always special. The formula for the virtual dimension gives:

$$v(2E) = 2v(E) + E^2$$

which means that $v(2E) = -1$ but $|2E|$ is not empty.

Definition 3.2. An irreducible and reduced curve E which satisfies the equations $E^2 = -1$ and $v(E) = 0$ is called a (-1)-curve.

Remark 3.3. As observed in the preceding example (-1)-curves are related to the construction of some special linear systems. One important fact to observe here is that the numerical conditions $E^2 = -1$ and $v(E) = 0$ are not enough to define a (-1)-curve. For example consider the system $\mathcal{L} = \mathcal{L}_2(5; 3^2, 1^8)$ which is not empty since its virtual dimension is 0. Let $C \in \mathcal{L}$ be a curve of the system, then $v(C) = 0$ and $C^2 = -1$ but it is easy to see that C is reducible since $CR = -1$, where R is a line through the two triple points. By corollary 2.3 we can conclude that R is a fixed component of $\mathcal{L}$. The difference $\mathcal{L} - R$ is the system $\mathcal{L}_2(4; 2^2, 1^8)$ which represents a quartic curve with two double points. We will prove later that this system is irreducible of dimension 0, so it contains only one curve B. Observe that $RB = 0$ so that C is not only reducible but also not connected.

Remark 3.4. The intersection form can be naturally extended to intersect the points p_i. In fact to any p_i one can associate the "system" $\mathcal{L}_2(0; 0, \ldots, -1, \ldots, 0)$, with the -1 at the i-th position. This implies that $p_i^2 = -1$ and $p_i K = -1$ so that the point p_i has the same numerical properties of a (-1)-curve. Moreover this extension of the intersection form immediately shows that its signature is $(1, r)$.

Remark 3.5. A classical way to construct a (-1)-curve is to consider the closure of the graph of the function $f : \mathbb{C}^2 \setminus \{(0,0)\} \to \mathbb{P}^1$ defined by $f(x,y) = [x : y]$. The closed graph of f is defined by $\Gamma_f := \{(x,y) \times [t_0 : t_1] \in \mathbb{C}^2 \times \mathbb{P}^1 \mid xt_1 = yt_0\}$ and the map

$$\pi : \Gamma_f \to \mathbb{C}^2$$

given by $\pi(x,y,[t_0 : t_1]) = (x,y)$ is called the blowing-up map of $\mathbb{C}^2$ along $(0,0)$. The curve $E := \pi^{-1}(0,0)$ is a smooth rational curve which takes the name of (-1)-curve.

4. Two conjectures

We are now ready to formulate one of the main conjecture about the structure of special linear systems of $\mathbb{P}^2$.

Conjecture (Segre)**.** *A special linear system $\mathcal{L}$ through multiple points in general position has a multiple fixed component.*

This means that for any two curves $C, C' \in \mathcal{L}$ the intersection $C \cap C' \supseteq mE$ where E is an irreducible and reduced curve and $m \geq 2$. In other word, if $P_C(x_0,x_1,x_2)$ is a defining polynomial for C, then

$$P_C(x_0,x_1,x_2) = P_E(x_0,x_1,x_2)^m P_{C-mE}(x_0,x_1,x_2),$$

where $P_E(x_0,x_1,x_2)^m$ is a common factor of all the polynomials associated to the curves in $\mathcal{L}$. The equation $P_E(x_0,x_1,x_2) = 0$ defines a curve E which is reduced.

It is possible to characterize E by means of the intersection form already introduced. First of all we rewrite the preceding equality with the following notation:

$$\mathcal{L} = mE + (\mathcal{L} - mE)$$

and observe that $\dim(\mathcal{L}) = \dim(\mathcal{L} - mE)$ since the correspondence between the curves in $\mathcal{L}$ and $\mathcal{L} - mE$ is a bijection.
Consider now the following formula obtained by 2

$$v(\mathcal{L} - E) = v(\mathcal{L}) + v(E) - E\mathcal{L}.$$

Since E is a linear system without multiple components, then the Segre conjecture says that $|E|$ is non-special. But E is in the fixed locus of $\mathcal{L}$, hence $\dim|E| = 0$ and so $v(E) = 0$. What we want to prove is that the preceding formula implies that $E\mathcal{L} < 0$. Suppose the contrary, then $v(\mathcal{L} - E) \leq v(\mathcal{L})$ and in the same way one has that $v(\mathcal{L} - mE) \leq v(\mathcal{L} - (m-1)E) \leq \ldots \leq v(\mathcal{L})$. But by hypothesis $\mathcal{L}$ is special, hence

$$v(\mathcal{L} - mE) \leq v(\mathcal{L}) < \dim \mathcal{L} = \dim(\mathcal{L} - mE).$$

So that also $v(\mathcal{L} - mE)$ is a special system which does not contain multiple components and this cannot happen if we assume that Segre conjecture is true.

The curve E is reduced but not necessarily irreducible, hence we can write $E = \sum E_i$ where the E_i are irreducible curves and consider the inequality

$$E(mE + \mathcal{M}) < 0,$$

where $\mathcal{M} = \mathcal{L} - mE$. Since $\mathcal{M}$ does not contain fixed components then $E_i\mathcal{M} \geq 0$ by corollary 2.3 and this means that $E^2 < 0$. By the same corollary we have that $E_iE_j \geq 0$ so that

$$E_i^2 < 0$$

for at least one E_i. Since E_i is irreducible and reduced then, by assuming Segre conjecture to be true, the system $|E_i|$ is non-special of dimension 0 (by corollary 2.3). Then $v(E_i) = 0$ and E_i is a (-1)-curve.

The preceding considerations show that Segre conjecture implies that a special linear system $\mathcal{L}$ contains always a multiple of a (-1)-curve as a fixed component.

Exercise 4.1. Prove that Segre conjecture implies actually that if $\mathcal{L}$ is special then there exists a (-1)-curve E_j such that $E_j\mathcal{L} \leq -2$.

We are now ready to state the following:

Conjecture (Harbourne-Hirschowitz). *A linear system $\mathcal{L}$ of $\mathbb{P}^2$ is special if and only if there exists a (-1)-curve E such that $E\mathcal{L} \leq -2$.*